Rotation Matrices

Linear Algebra

Major sources: Richard Murray,
Zexiang Li,
S. Shankar Sastry,
Aaron Bestick

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Rotation Matrices

Special Orthogonal Group: $\mathbf{SO}(n)$ Lie group

$R^T R = RR^T = I$ …columns & rows orthonormal $\iff$ $R^{-1} = R^T$

$\det(R) = 1$ …no reflections

$R \in \mathrm{SO}(2) \subset \mathbb{R}^{2\times 2}$

Diagonalization:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\frac{1}{\sqrt{2}}$$

right evecs **left evecs**

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad U^*U = UU^* = I$$

…columns orthonormal

$$R^T R = \begin{bmatrix} - & R_1^T & - \\ & \vdots & \\ - & R_n^T & - \end{bmatrix}\begin{bmatrix} | & & | \\ R_1 & \cdots & R_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} R_1^T R_1 & \cdots & R_1^T R_n \\ \vdots & & \vdots \\ R_n^T R_1 & \cdots & R_n^T R_1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

…similar for rows

$$RR^T = \begin{bmatrix} - & r_1^T & - \\ & \vdots & \\ - & r_n^T & - \end{bmatrix}\begin{bmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T r_1 & \cdots & r_1^T r_n \\ \vdots & & \vdots \\ r_n^T r_1 & \cdots & r_n^T r_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Rotation Matrices - 2D

Special Orthogonal Group: $\mathbf{SO}(n)$ Lie group

$$R^T R = R R^T = I$$ …columns & rows orthonormal

$$\det(R) = 1$$ …no reflections

$$R \in \mathrm{SO}(2) \subset \mathbb{R}^{2\times 2}$$

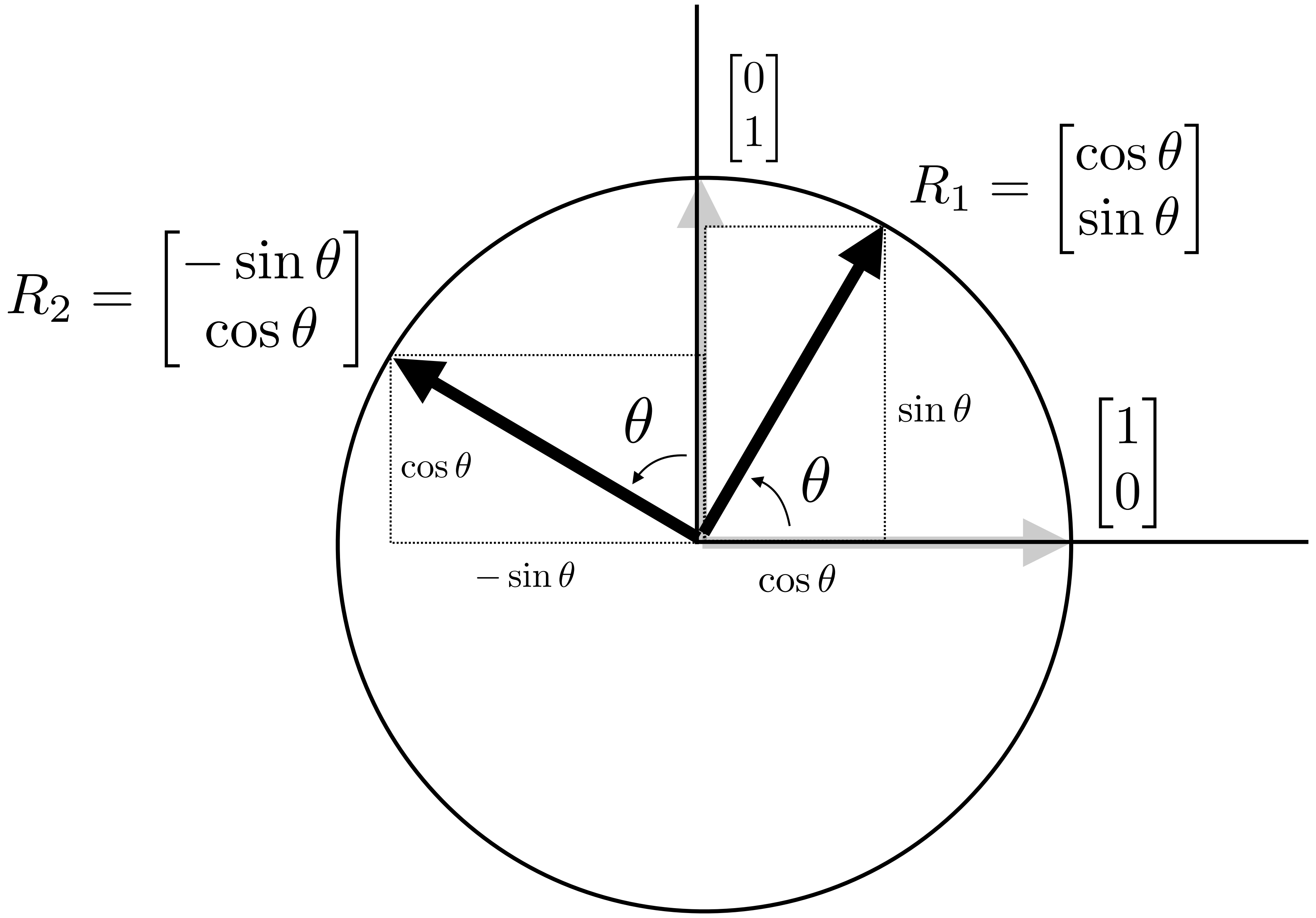
Diagonalization:

$$R = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix}\begin{bmatrix}e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{bmatrix}\begin{bmatrix}1 & -i \\ 1 & i\end{bmatrix}\frac{1}{\sqrt{2}}$$

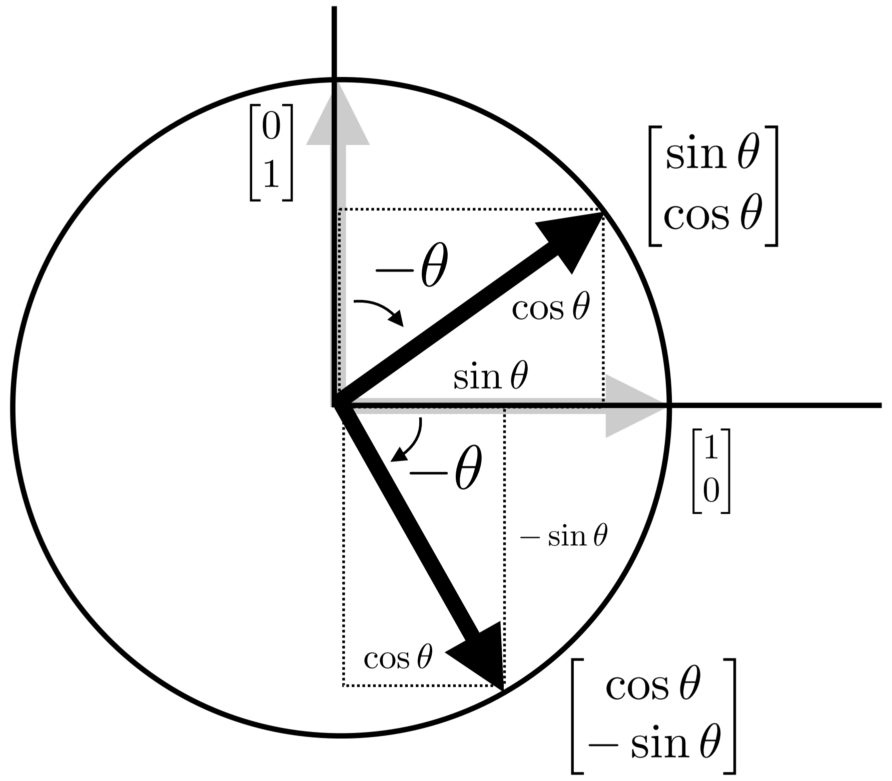
right evecs **left evecs**

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix} \qquad U^*U = UU^* = I$$

$$R = \begin{bmatrix} R_1 & R_2 \end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$



$$R^{-1} = R^T = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}$$



Rotation Matrices - 3D

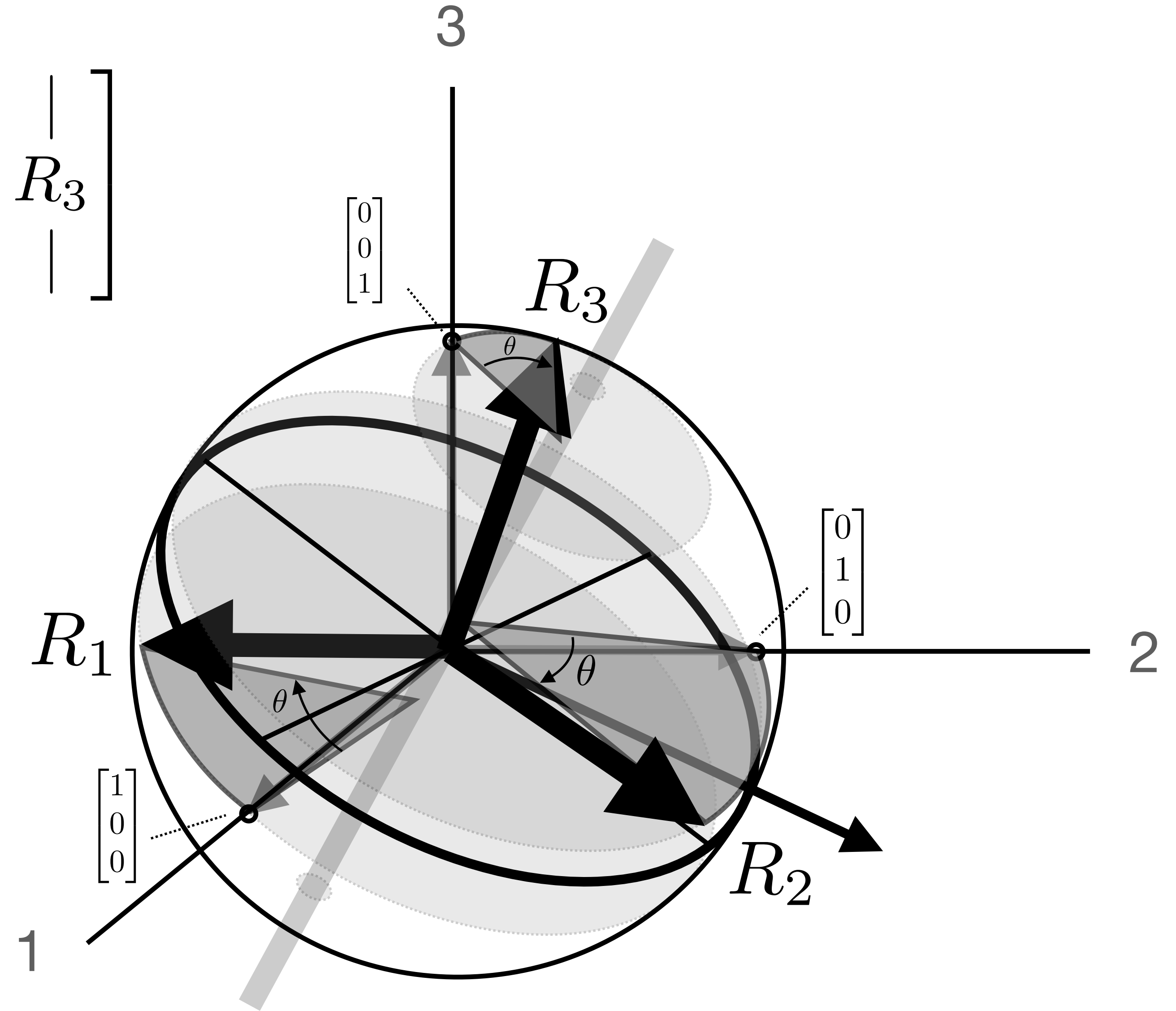
Special Orthogonal Group: $\mathbf{SO}(n)$ Lie group

$R^T R = R R^T = I$ ...columns & rows orthonormal

$\det(R) = 1$ ...no reflections

$R \in \mathrm{SO}(3) \subset \mathbb{R}^{3\times 3}$

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

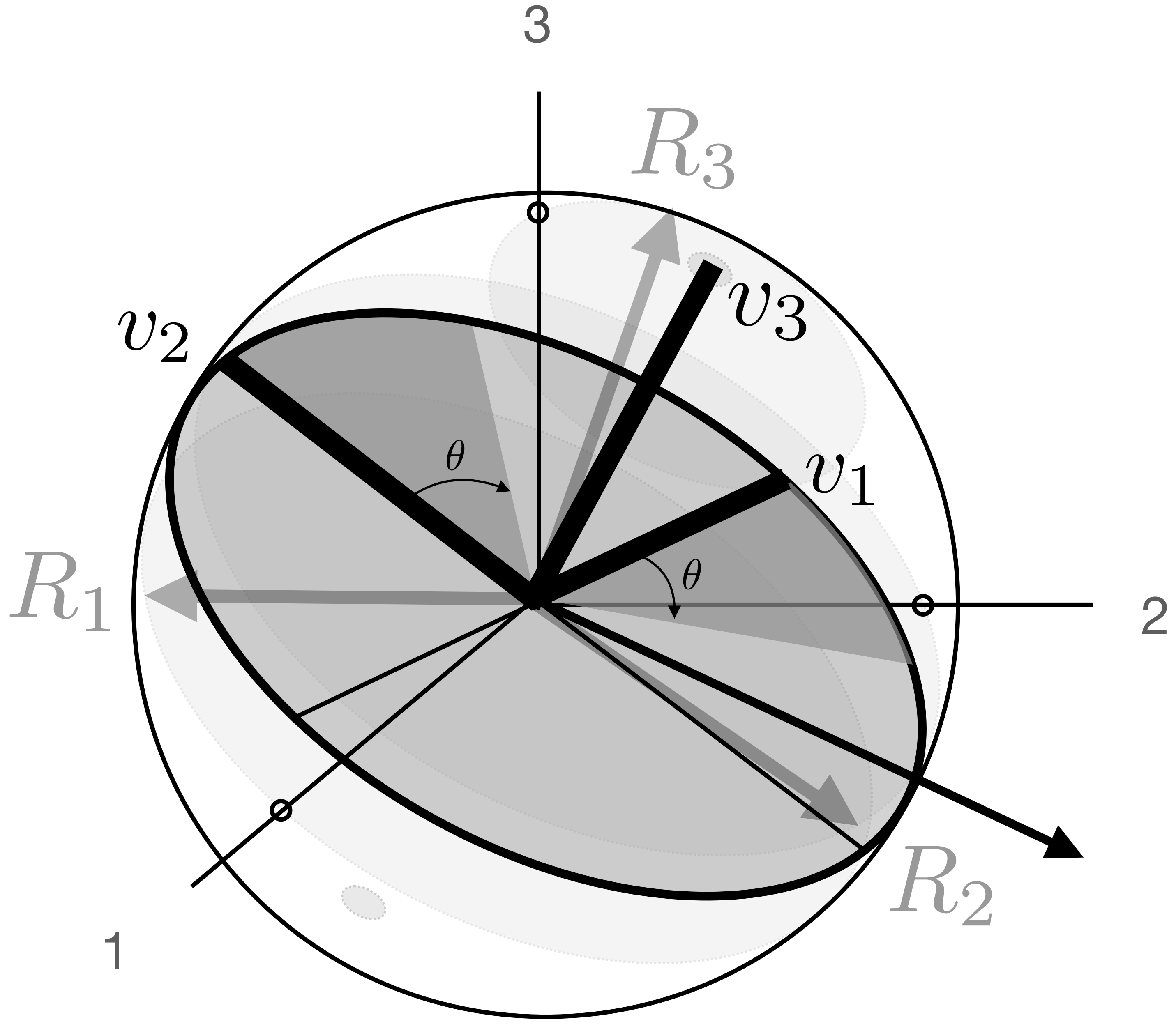


Diagonalization:

$$R = \begin{bmatrix} | & | & | \\ \frac{1}{\sqrt{2}}(v_1 + i v_2) & \frac{1}{\sqrt{2}}(v_1 - i v_2) & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(v_1^T - i v_2^T) & - \\ - & \frac{1}{\sqrt{2}}(v_1^T + i v_2^T) & - \\ - & v_3^T & - \end{bmatrix}$$

U **right eigenvectors** U^* **left eigenvectors**

unitary: $U^* U = U U^* = I$



Rotation plane **Rotation axis**

$$R = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3^T & - \end{bmatrix}$$

orthonormal: V $V^T V = V V^T = I$ V^T

Rotation Matrices - 3D

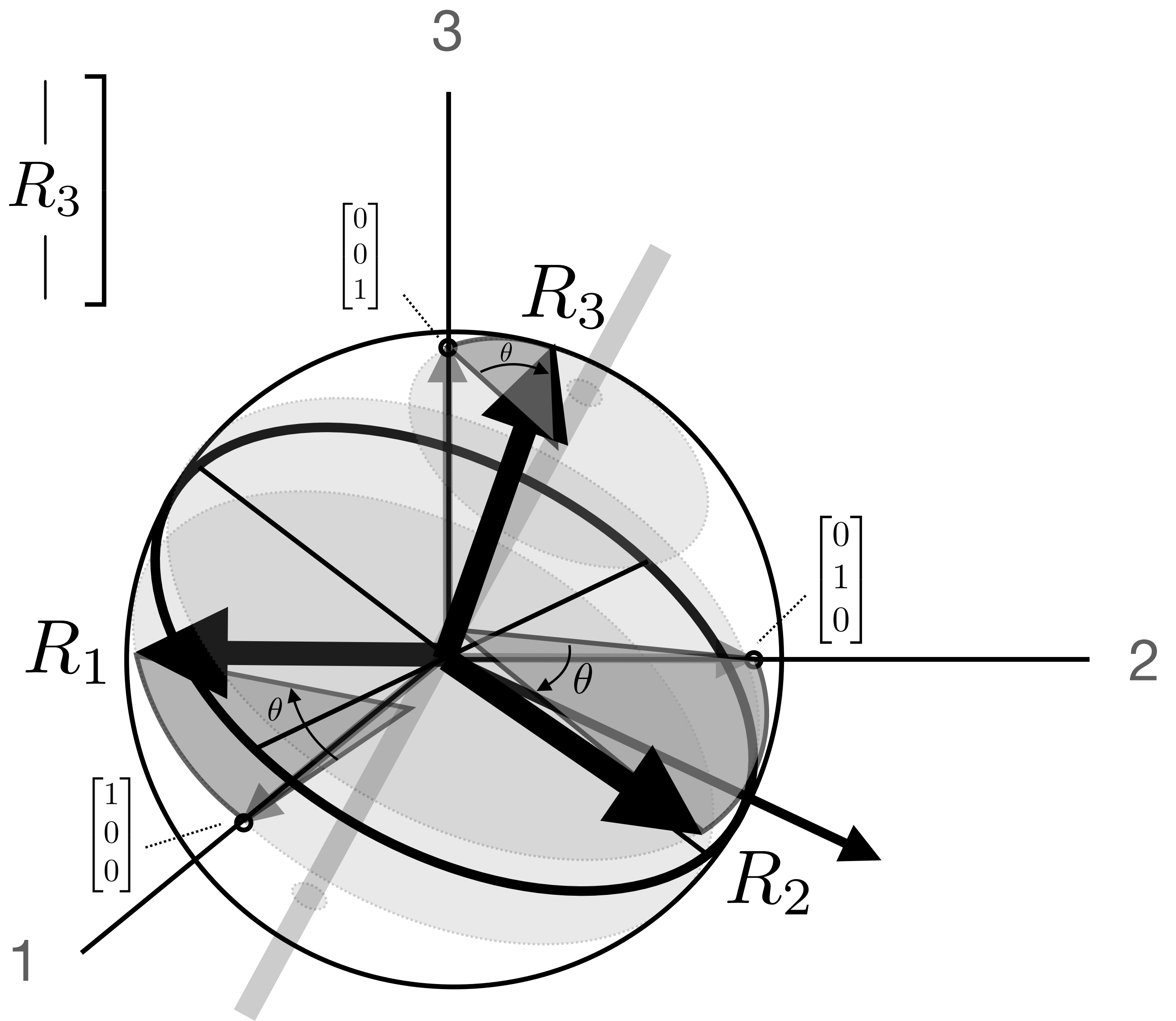
Special Orthogonal Group: $\mathbf{SO}(n)$ Lie group

$$R^T R = R R^T = I$$ ...columns & rows orthonormal

$$\det(R) = 1$$ ...no reflections

$$R \in \mathrm{SO}(3) \subset \mathbb{R}^{3\times 3}$$

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$



Diagonalization:

$$R = \begin{bmatrix} | & | & | \\ \frac{1}{\sqrt{2}}(v_1 + iv_2) & \frac{1}{\sqrt{2}}(v_1 - iv_2) & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(v_1^T - iv_2^T) & - \\ - & \frac{1}{\sqrt{2}}(v_1^T + iv_2^T) & - \\ - & v_3^T & - \end{bmatrix}$$

U **right eigenvectors** U^* **left eigenvectors**

unitary: $U^* U = U U^* = I$

Rotation plane **Rotation axis**

$$R = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3^T & - \end{bmatrix}$$

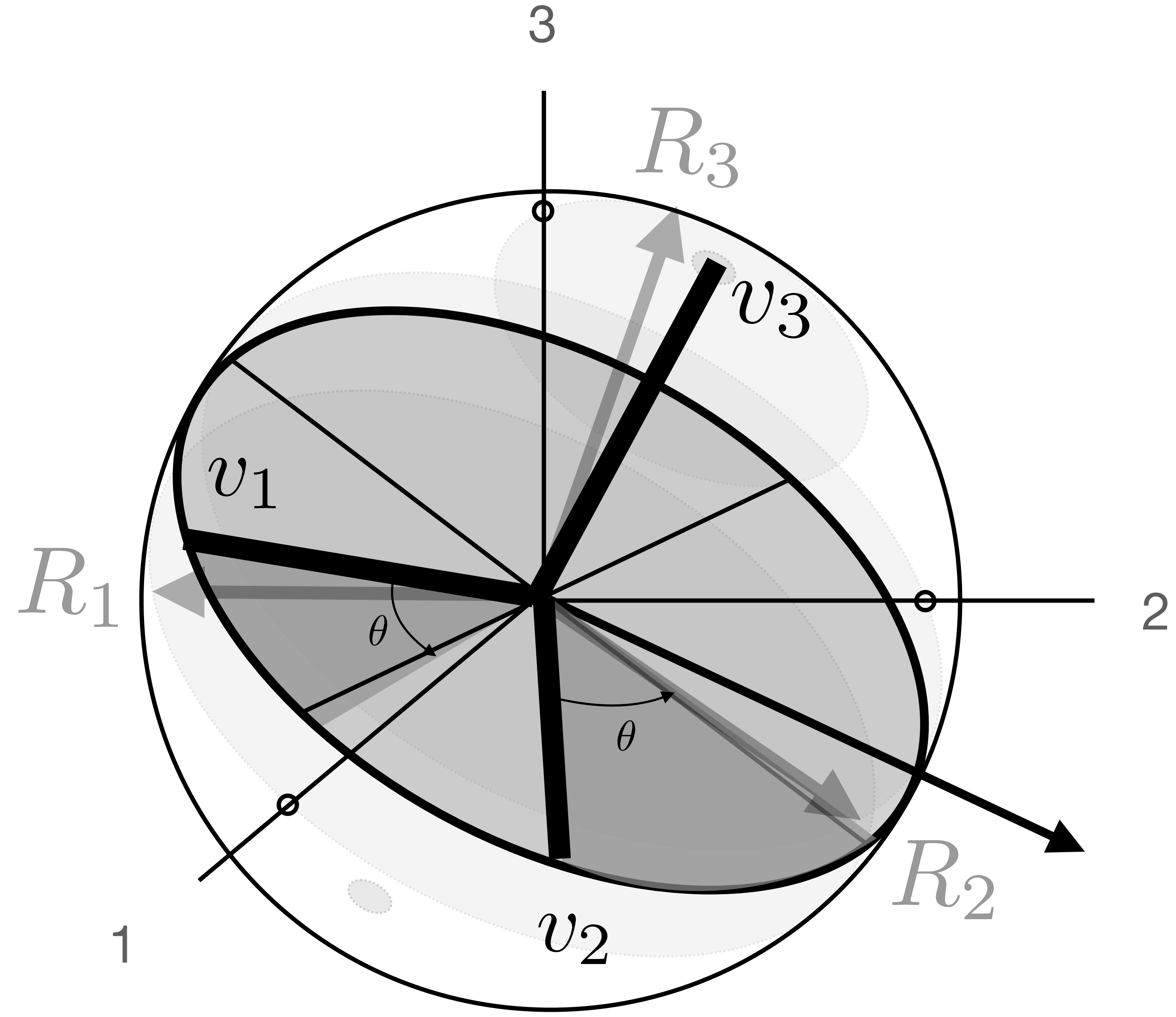
orthonormal: V $V^T V = V V^T = I$ V^T

Note: can change v_1, v_2 by a phase shift ϕ

$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \Leftarrow \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix}$$

...since

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} = \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\phi & s\phi \\ -s\phi & c\phi \end{bmatrix}$$



Skew symmetric matrices

Special orthogonal Lie algebra $\mathbf{so}(n)$ Lie algebra

$$K = -K^T$$ ...skew symmetric matrix $K \in \mathrm{so}(n) \subset \mathbb{R}^{n\times n}$

$k \in \mathbb{R}^3 \quad ||k||_2 = 1$ ...axis of rotation

$$K = \hat{k} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$ ...hat operator

$$R = \begin{bmatrix} | & | & | \\ R_1 & R_2 & R_3 \\ | & | & | \end{bmatrix}$$

$$R = e^{K\theta}$$ ...matrix exponential

Diagonalization:

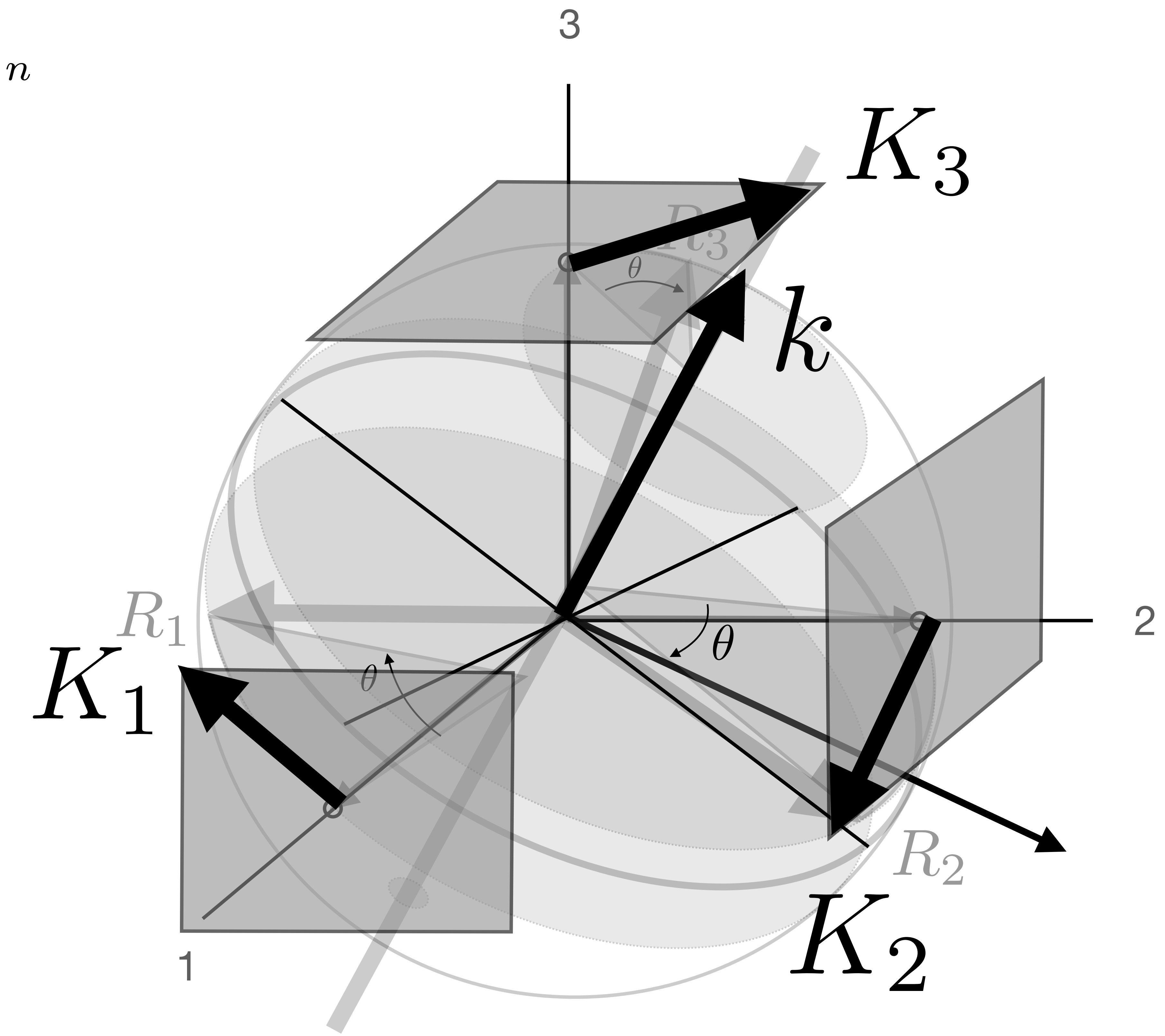
$$K\theta = \begin{bmatrix} | & & | \\ \frac{1}{\sqrt{2}}(v_1 + iv_2) & \frac{1}{\sqrt{2}}(v_1 - iv_2) & k \\ | & & | \end{bmatrix}\begin{bmatrix} -i\theta & 0 & 0 \\ 0 & i\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} - & \frac{1}{\sqrt{2}}(v_1^T - iv_2^T) & - \\ - & \frac{1}{\sqrt{2}}(v_1^T + iv_2^T) & - \\ - & k^T & - \end{bmatrix}$$

right eigenvectors **left eigenvectors**

Rotation plane **Rotation axis**

$$K\theta = \begin{bmatrix} | & | & | \\ v_1 & v_2 & k \\ | & | & | \end{bmatrix}\begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & k^T & - \end{bmatrix}$$

orthonormal: $V \qquad V^T V = V V^T = I \qquad V^T$



$$K = \begin{bmatrix} | & | & | \\ K_1 & K_2 & K_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$