Univ. of Washington

Lecture : Linear & Convex Combinations

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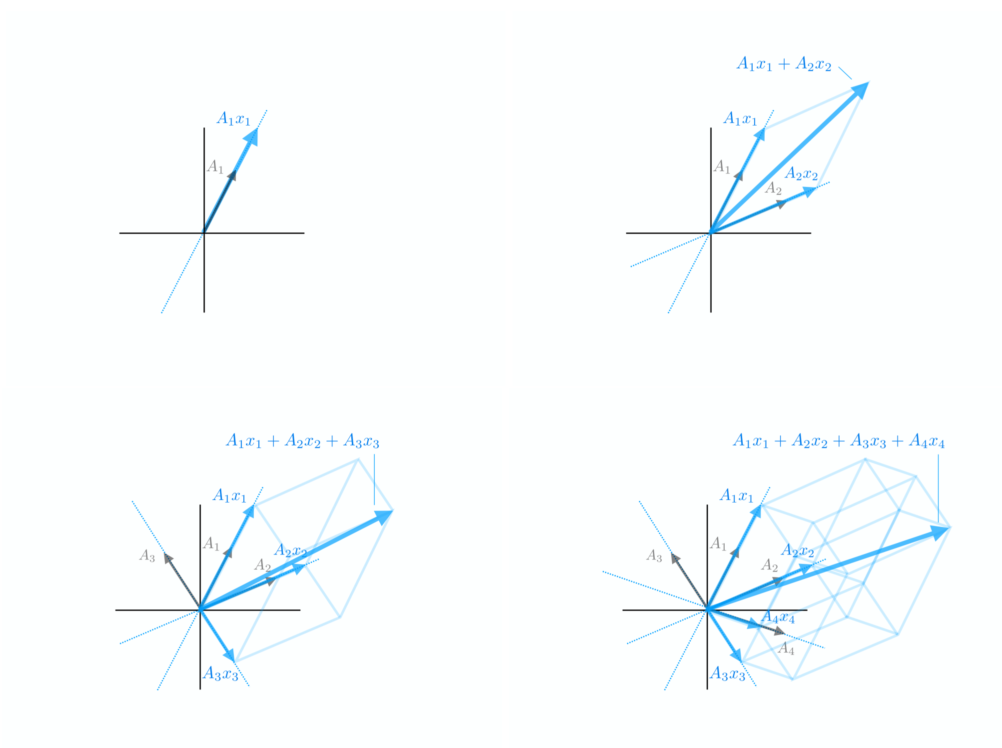
Linear Combinations

A linear combination of a set of vectors $\{A_1, \ldots, A_n\}$ is weighted sum of those vectors

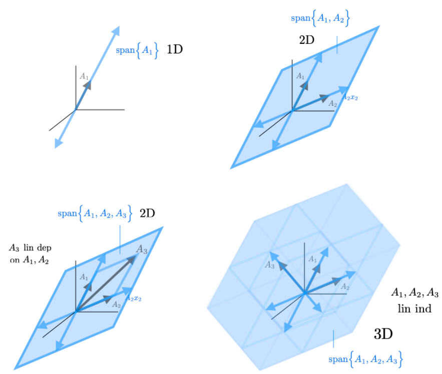
$$A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$$

where the weights or coefficients are scalars $x_1, \ldots, x_n \in \mathbb{R}$. Basic linear combinations of one, two, three, and four vectors are shown below.



Span

The set of all possible linear combinations is called the span of a set of vectors. Spans of one,two, and three vectors are shown in the image below. Note that spatially The span of a set of vectors forms a hyperplane or subspace that passes through the origin and extends out in any directions the vectors point. Since negative coefficients are possible the span also extends in the opposite direction of the vectors. If we add a new vector to the set of vectors that does not already lie in the span of the original vectors then we increase the dimension of the spanned subspace. If we add a new vector that was already in the span, the dimension of the spanned subspace does not increase

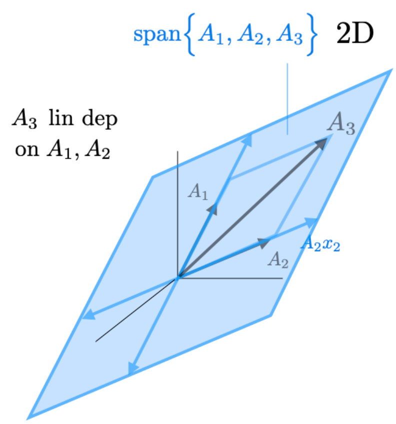


Linear Dependence

We say a vector is linearly dependent on a set of vectors if it lies in the span of those vectors, ie. one can construct that vector as a linear combination of vectors in the set. Algebraically, y is linearly dependent on a set of vectors $\{A_1, \ldots, A_n\}$, we can find coefficients $x_1, \ldots, x_n \in \mathbb{R}$ such that

$$y = A_1x_1 + \cdots + A_nx_n$$

If a vector is not linearly dependent on a set of vectors, we say it is linearly independent from that set. We say a set of vectors is linearly independent if none of the vectors are linearly dependent on the other vectors in the set, and we say the set is linear dependent if any of the vectors are linearly dependent on the others. The image below shows the case where a vector A_3 is linearly dependent on vectors $A_1, A_2 \in \mathbb{R}^3$.



More Math: Rigorous Definitions

This next section discusses linear dependence and independence in rigorous mathematical terms. A student of linear algebra will quickly note that understanding/working with these definitions can actually be a lot more difficult than understanding the concepts given above. While one should not shy away from the rigorous math, one should also keep in mind the simple intuition for the concepts given above.

A compact way to state mathematically that a set of vectors $\{A_1, \ldots, A_n\}$ is linearly dependent is to say, there exists a vector of coefficients $x \neq 0$ such that

$$A_1 x_1 + \cdots + A_n x_n = 0$$

This statement encodes that there is at least one vector A_i that is dependent on the others. Here, $x \neq 0$ means that at least one of the coefficients is not equal to 0. Since at least one x_i is nonzero (assume it is $i = 1$ for simplicity), we can write

$$A_1 x_1 = -A_2 x_2 - \cdots - A_n x_n$$

We can then explicitly write A_1 as a linear combination of the others

$$A_1 = A_2 \left(-\tfrac{x_2}{x_1}\right) + \cdots + A_n \left(-\tfrac{x_n}{x_1}\right)$$

Note: if all the other x_i's are zero, then A_1 must be the zero vector which is linear dependent on any set of vectors. Negating the above statement gives a mathematical definition of linear independence. A set of vectors $\{A_1, \ldots, A_n\}$ is linearly independent if there does not exist a nonzero vector x such that $A_1 x_1 + \cdots + A_n x_n = 0$. We can rephrase this in several ways: a set is linearly independent if $A_1 x_1 + \cdots + A_n x_n = 0$ only when $x = 0$ or a set of vectors $\{A_1, \ldots, A_n\}$ is linearly independent if

$$A_1 x_1 + \cdots + A_n x_n = 0 \qquad \Rightarrow \qquad x = 0$$

This last characterization is by far the most useful in mathematical proofs. If we can show that the sum condition on the left implies $x = 0$ then we know the set of vectors is linearly independent.

More Math: Proving Linear Independence (Difficulty: 3/10)

These next comments assume an understanding of matrix multiplication and matrix column geometry.

Practically when we try to prove linear independence (or dependence) of a set of vectors (say n vectors each in $\mathbb{R}^m$), we often write them as columns of a matrix $A \in \mathbb{R}^{m \times n}$ and write vector of coefficients as $x \in \mathbb{R}^n$. The linear dependence condition becomes there exist $x \neq 0$ such that $Ax = 0$, ie. A has a nontrivial right nullspace. The linear independence condition becomes $Ax = 0 \Rightarrow x = 0$. This is quite compact and useful. For example, suppose A can be divided into rows A' and A'' where we already know that A' has linearly independent columns. We can then show immediately that A must have linearly independent columns as well

$$Ax = \begin{bmatrix} A' \\ A'' \end{bmatrix} x = \begin{bmatrix} A'x \\ A''x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $A'x = 0$ then $x = 0$ (by the linear independence of the columns of A') and thus we have shown that $Ax = 0$ implies $x = 0$ as desired. This often arises in the even simpler context where A' is just the identity matrix. In this case we simply have

$$Ax = \begin{bmatrix} I \\ A'' \end{bmatrix} x = \begin{bmatrix} x \\ A''x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x = 0$$

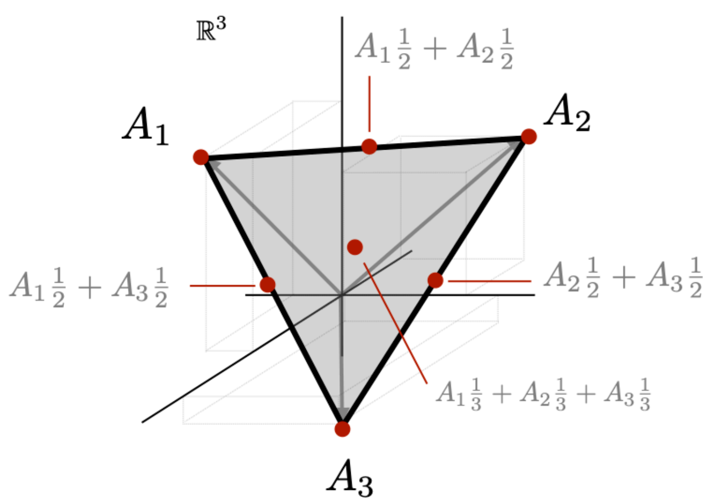
We will return to this construction in our construction of bases for range and nullspaces.

Convex Combinations

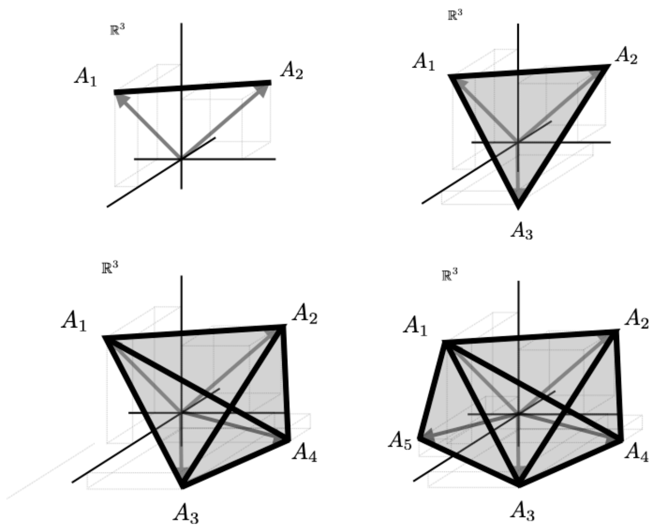
One specific type of linear combination that often arises is a convex combination A convex combination of vectors is formed when we require the coefficients of a linear combination to all be positive and sum to one, ie.

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n, \qquad \text{with} \qquad \sum_i x_i = 1, \qquad x_i \geq 0$$

Note that we also write this constraint in more compact notation as $\mathbf{1}^T x = 1,\ x \geq 0$ or even more simply as $x \in \Delta_n$ where Δ_n is the nD simplex. Visually, a convex combination of vectors live in the space between the vectors or the convex hull. For any two vectors A_1 and A_2, $A_1\frac{1}{2} + A_2\frac{1}{2}$ is the vector halfway between the two vectors; for n vectors $\sum_i A_i \frac{1}{n}$ is the center of the set of vectors ("average" vector of the set); etc. Different convex combinations of vectors $\{A_1, A_2, A_3\} \in \mathbb{R}^3$ are illustrated in the figures below.



We can also show convex combinations of two, three, four, and five vectors.



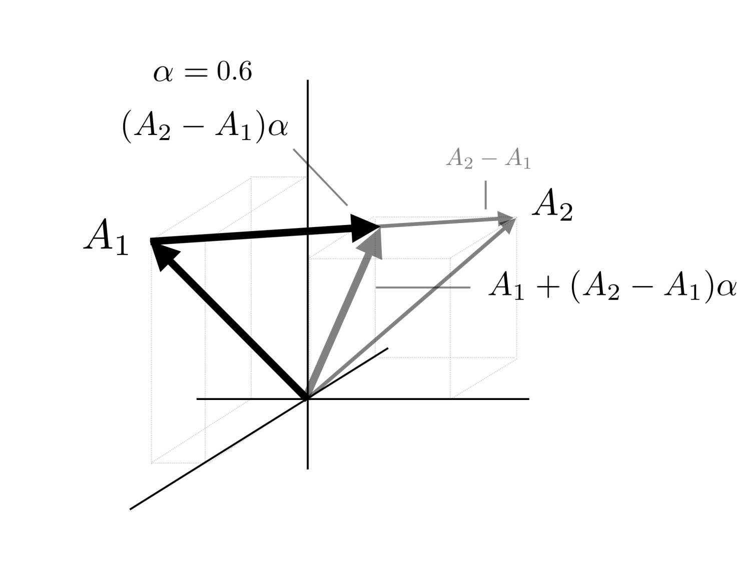
A convex combination between just two vectors is often written in the form

$$y = A_1(1 - \alpha) + A_2\alpha$$

with $0 \leq \alpha \leq 1$. We notice that if we start out with $\alpha = 0$, $y = A_1$. We note that we can rearrange the above expression to show explicitly how y changes with α

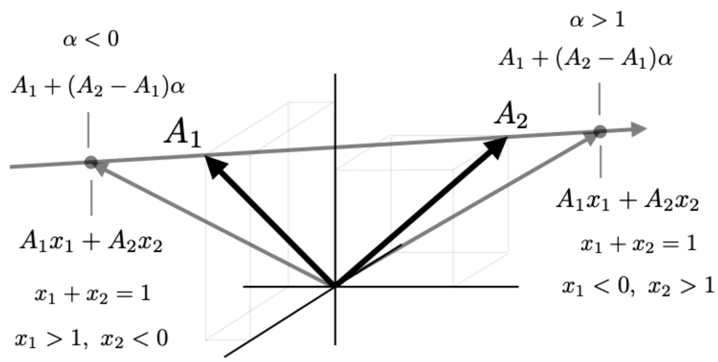
$$y = A_1 + (A_2 - A_1)\alpha$$

As $\alpha moves from 0 to 1, y$ starts at A_1 and then adds back little bits of the difference from A_1 to A_2 until y reaches A_2. This perspective is illustrated in the figure below.

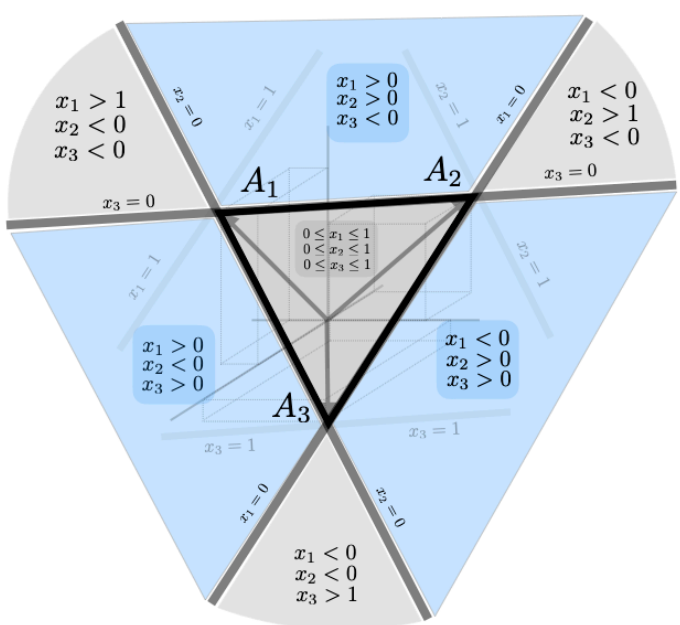


Keeping α between 0 and 1 keeps y on the segment between A_1 and A_2. If we relax this condition and allow α to be negative or greater than 1, we trace out a whole line that runs through A_1 and A_2. $\alpha < 0$ extends past A_1; $\alpha > 1$ extends off past A_2.

Note that here $x_1 = (1 - \alpha)$ and $x_2 = \alpha$. Removing the restriction on α is equivalent to removing the restriction that $x_1 \geq 0$ and $x_2 \geq 0$. By construction, we still have $x_1 + x_2 = (1 - \alpha) + \alpha = 1$



Note that this extension to the line through A_1 and A_2 applies to larger numbers of vectors as well. For example for three vectors $\{A_1, A_2, A_3\}$, removing the restriction that $x \geq 0$ expands the convex set to the plane shown below. Note the signs of x_1,x_2, and x_3 in each part of the plane.



More Drawing: Is a Point Inside a Triangle? (Difficulty: 3/10)

The above construction can be quite useful for determining quickly if a point is inside a triangle, tetrahedron, or hyper-tetrahedron. (This has many applications. For example it is a critical step in the implementation of the quickhull algorithm for computing convex hulls of points.) We focus on the case of three points in 2D, ie. determining if a point is inside a triangle. To determine if a point $y \in \mathbb{R}^2$ is inside a triangle formed by points $\{A_1, A_2, A_3\} \subset \mathbb{R}^2$, we can form the following system of equations

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \\ - & \mathbf{1}^T & - \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 1 & 1 & 1 \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The last row this system of equations requires x to sum to 1, ie. x generates linear combinations in the plane shown in the figure above. The first two rows then say that y is a linear combination of the A_i vectors. Note assuming this system is invertible, we can compute $x = M^{-1}y$. After computing x one can immediately read off which section of the plane y is in based on the signs of the elements of x. If $x \geq 0$, then y is in the triangle. For fast implementation a 3×3 matrix M can be inverted explicitly (see inverse formulas).

Note: this system will always be invertible unless the triangle shown above collapse to a line and the problem is ill-posed. One could still actually take the pseudo-inverse which would determine if the projection of y onto that line was inside the convex combination of the points. (I think.)

For tetrahedrons or hyper-tetrahedrons in nD, this formula extends to checking if $y \in \mathbb{R}^n$ is a convex combination of $n+1$ points $\{A_1, \ldots, A_{n+1}\}$. The equation then becomes

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ A_1 & \cdots & A_{n+1} \\ | & & | \\ - & \mathbf{1}^T & - \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix}$$

Taking $x = M^{-1}y$, one can similarly check that $x \geq 0$.

More Math: Coordinate Transformations on Convex Hulls (Difficulty: 4/10)

When we're working with a convex hull of a set of points, we may often want to write different coordinate systems based on those points. We list several useful coordinate transformations below along with explicit representations of their inverses, diagrams, and intuitive explanations. Of course, other useful transformations are possible. We write the formulas for points in $\mathbb{R}^n$, but we draw figures for points in $\mathbb{R}^3$ and $\mathbb{R}^4$. The convex hull for 3 points in $\mathbb{R}^3$ is a triangle; for 4 points in $\mathbb{R}^4$ the convex hull is a 3D tetrahedron which can visualized. The first $n-1$ coordinates in our tranformations will line up explicitly with each convex hull. The last coordinate will be related to the vector $\mathbf{1}$.