Univ. of Washington

Lecture : Complex Numbers

Winter 2021

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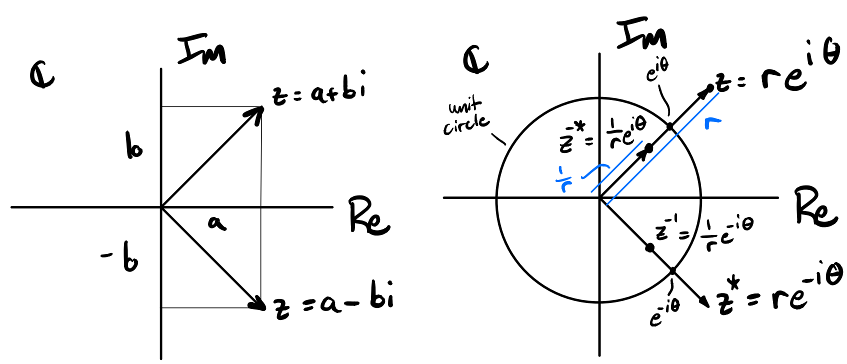
1 Complex Numbers

- Complex number: $z \in \mathbb{C}$.
- Cartesian representation: $z = a + bi$
  - Vector-like addition: $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$
  - Norm (length): $|z| = \sqrt{z^* z} = \sqrt{(a - bi)(a + bi)} = \sqrt{a^2 + b^2}$
  - Conjugate: $z^* = \bar{z} = a - bi$.
  - Inverse and Conjugate Inverse:

$$z^{-1} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a}{\sqrt{a^2+b^2}} + \frac{-b}{\sqrt{a^2+b^2}}i$$
$$z^{-*} = \bar{z}^{-1} = \frac{1}{a-bi} = \frac{a+bi}{(a-bi)(a+bi)} = \frac{a}{\sqrt{a^2+b^2}} + \frac{+b}{\sqrt{a^2+b^2}}i$$

  - Multiplication: $z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + (a_1 b_1 + a_2 b_2)i + b_1 b_2$.



- Polar representation: $z = re^{i\theta},\ r \geq 0$
  - Relationship to Cartesian representation:

$$z = a + bi = r\cos(\theta) + r\sin(\theta)i,$$
$$z = re^{i\theta} = \sqrt{z^* z}\, e^{i\tan^{-1}\left(\frac{a}{b}\right)} = \sqrt{a^2+b^2}\, e^{i\tan^{-1}\left(\frac{a}{b}\right)}$$

- Stretching and Rotation:
  The polar represents the stretching and rotational components of a complex number.

$$z = \underbrace{r}_{\substack{\text{Stretching}\\ \text{by } r}} \underbrace{e^{i\theta}}_{\substack{\text{Rotation}\\ \text{by } \theta}}$$

- Conjugate: $z^* = \bar{z} = re^{-i\theta}$.
- Inverse and Conjugate Inverse:

$$z^{-1} = \frac{1}{r}e^{-i\theta}$$
$$z^{-*} = \bar{z}^{-1} = \frac{1}{r}e^{i\theta}$$

- Multiplication: $z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

• Roots of Unity:

  - Solutions to the equation: $z^n = 1$.
  - n solutions:

$$z = e^{i\frac{2\pi k}{n}}, \quad \text{for} \quad k = 0, 1, 2, \ldots, n-2, n-1$$

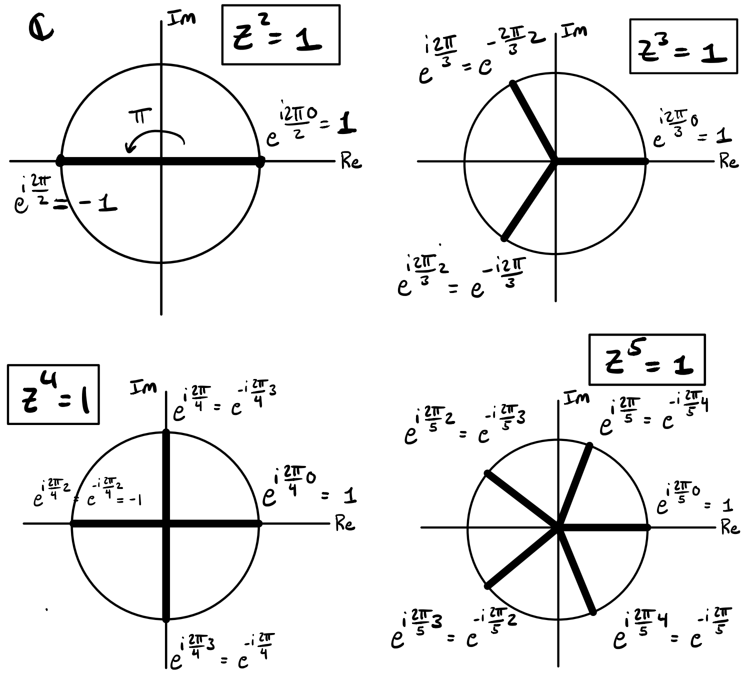
  - Each solution corresponds to an angle step size $\Delta\theta = \frac{2\pi k}{n}$ and powers of $z = e^{i\frac{2\pi k}{n}}$ represent stepping around the circle. k corresponds to the number of rotations around the unit circle before returning to 1. $k = 0$ is zero rotations, $k = 1$ is one rotation, $k = 2$ is two rotations, etc.
  - Alternative enumeration of solutions corresponding to rotating in reverse:

$$z = e^{i\frac{2\pi(-k')}{n}}, \quad \text{for} \quad k' = n, (n-1), \ldots, 2, 1$$

  by the relationship $k = n - k'$

$$z = e^{i\frac{2\pi(-k')}{n}} = e^{i\frac{2\pi(-k')}{n}} e^{i\frac{2\pi n}{n}} = e^{i\frac{2\pi(n-k')}{n}} = e^{i\frac{2\pi k}{n}}$$

  Pairs: $k = (n-1)$ and $-k' = -1$, $k = (n-2)$ and $-k' = -2$, etc.



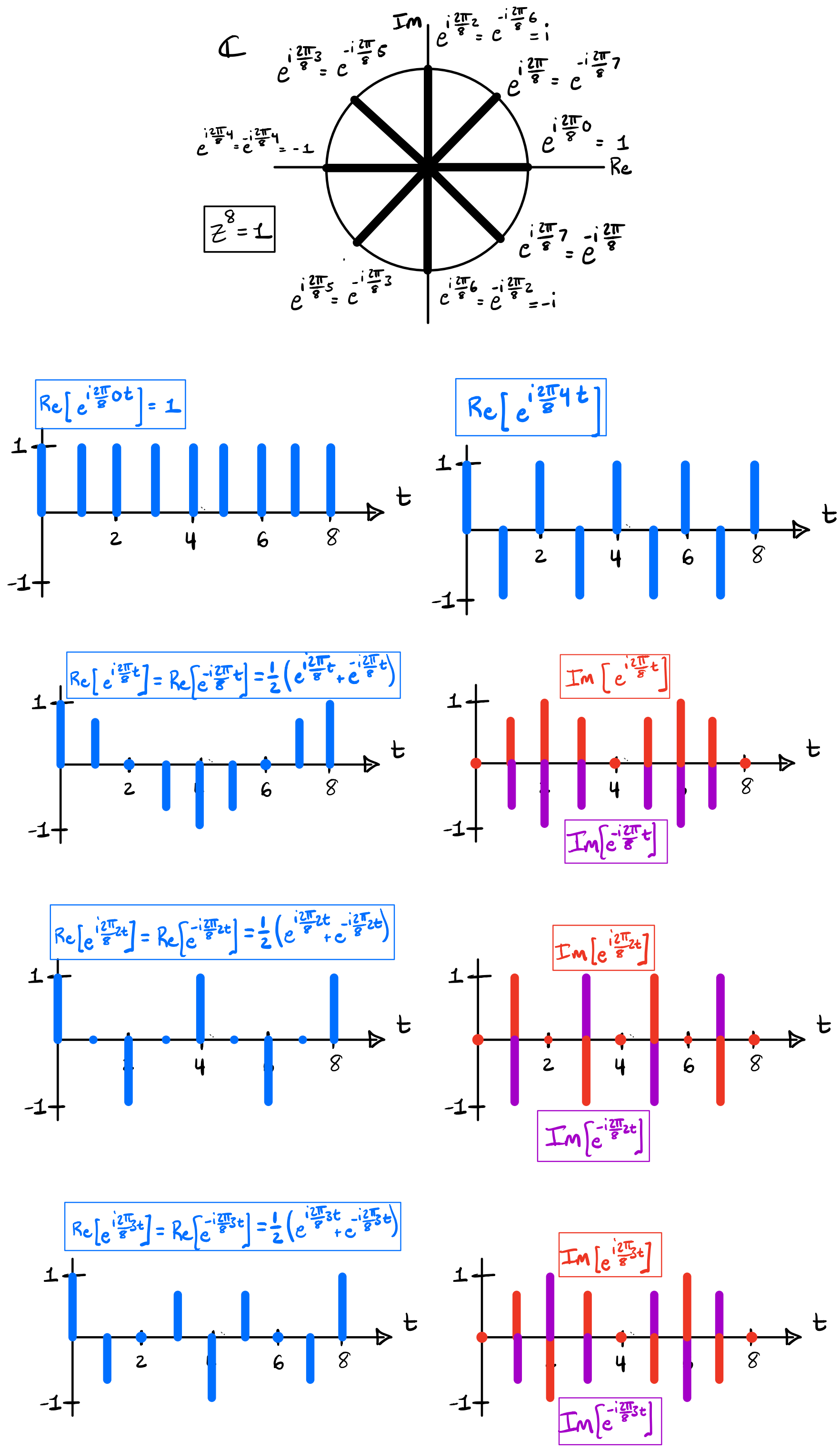
– Roots of unity can be used to define oscillating signals in discrete time.

  Let $F^k \in \mathbb{C}^n$ be defined as $\left[F^k\right]_t = e^{\left(i\frac{2\pi k}{n}\right)t}$, ie.

$$F^k = \begin{bmatrix} e^{\left(i\frac{2\pi k}{n}\right)0} & e^{\left(i\frac{2\pi k}{n}\right)1} & \dots & e^{\left(i\frac{2\pi k}{n}\right)(n-1)} \end{bmatrix}^T$$

  In discrete time Fourier analysis, we often use the matrix DFT (discrete Fourier transform) matrix $F \in \mathbb{C}^{n\times n}$.

$$\begin{aligned}
F &= \begin{bmatrix} F^0 & F^1 & \cdots & F^{n-1} \end{bmatrix} \\
&= \begin{bmatrix}
e^{\left(i\frac{2\pi 0\times 0}{n}\right)} & e^{\left(i\frac{2\pi 0\times 1}{n}\right)} & \dots & e^{\left(i\frac{2\pi 0\times (n-1)}{n}\right)} \\
e^{\left(i\frac{2\pi 1\times 0}{n}\right)} & e^{\left(i\frac{2\pi 1\times 1}{n}\right)} & \dots & e^{\left(i\frac{2\pi 1\times (n-1)}{n}\right)} \\
\vdots & \vdots & \ddots & \vdots \\
e^{\left(i\frac{2\pi (n-1)\times 0}{n}\right)} & e^{\left(i\frac{2\pi (n-1)\times 1}{n}\right)} & \dots & e^{\left(i\frac{2\pi (n-1)\times (n-1)}{n}\right)}
\end{bmatrix} \\
&= \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & e^{\left(i\frac{2\pi 1\times 1}{n}\right)} & \dots & e^{\left(i\frac{2\pi 1\times (n-1)}{n}\right)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & e^{\left(i\frac{2\pi (n-1)\times 1}{n}\right)} & \dots & e^{\left(i\frac{2\pi (n-1)\times (n-1)}{n}\right)}
\end{bmatrix}
\end{aligned}$$



The columns of F can be used to represent time oscillating signals. In discrete time

Fourier analysis a time (or phase) shift of $\frac{k}{n}$ Hz can be represented by the root of unity $e^{\left(i\frac{2\pi k}{n}\right)}$. Multiplying F^k by $e^{\left(i\frac{2\pi k}{n}\right)}$ shifts each element of the vector up one spot and moves the first element to the end.

$$\begin{aligned} e^{\left(i\frac{2\pi k}{n}\right)}F^k &= e^{\left(i\frac{2\pi k}{n}\right)}\left[e^{\left(i\frac{2\pi k}{n}\right)0} \quad e^{\left(i\frac{2\pi k}{n}\right)1} \quad \cdots \quad e^{\left(i\frac{2\pi k}{n}\right)(n-1)}\right]^T \\ &= \left[e^{\left(i\frac{2\pi k}{n}\right)1} \quad e^{\left(i\frac{2\pi k}{n}\right)2} \quad \cdots \quad e^{\left(i\frac{2\pi k}{n}\right)(n-1)} \quad e^{\left(i\frac{2\pi k}{n}\right)0}\right]^T \end{aligned}$$