Univ. of Washington

Lecture : Bases and Coordinates

Winter 2021

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Basis

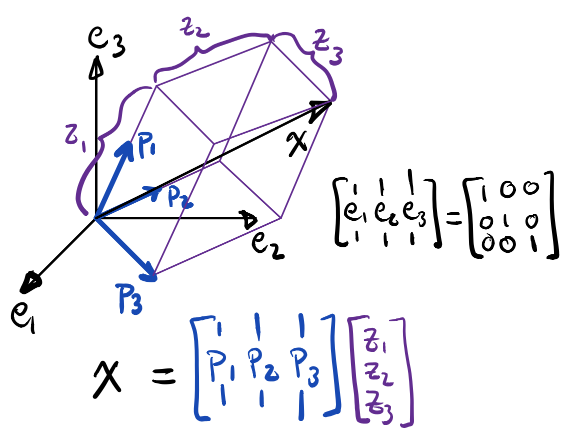
A set of vectors $\{V_i\}_{i=1}^n$ are a **basis** for a vector space $\mathcal{V}$ if

- $\{V_i\}_{i=1}^n$ span $\mathcal{V}$.
- $\{V_i\}_{i=1}^n$ are linearly independent

Every basis for $\mathcal{V}$ has the same number of vectors and the number of vectors in a basis for $\mathcal{V}$ is called the **dimension** of $\mathcal{V}$.

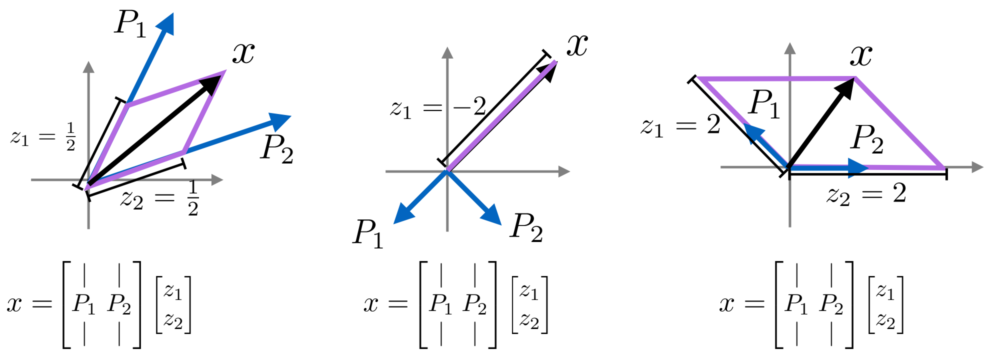
Change of Basis

Suppose we have a basis for $\mathbb{R}^n$ stored in the columns of a square matrix $P \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$. The *coordinates* of x with respect to the basis P are the coefficients of the linear combination of the columns of P that gives the vector x. We represent these coordinates in a vector $z \in \mathbb{R}^n$ and have that $x = Pz$. Note that x is really the coordinates of itself with respect to the *standard basis vectors* which are the columns of the identity matrix, ie. $x = Ix$.



To compute the coordinates of x with respect to P, we simply invert P, ie. $z = P^{-1}x$. If the columns of P are a basis, then they must be linearly independent and P is invertible.

For simple 2×2 and 3×3 cases, we could also compute the coordinates z by drawing the basis vectors and x and eyeballing the appropriate coordinates as in the figure above. This is a useful exercise in order to develop intuition coordinates. We give several examples in the figures below. (Once you guess z, you can always easily check how close you are by computing Pz and comparing it with x.)

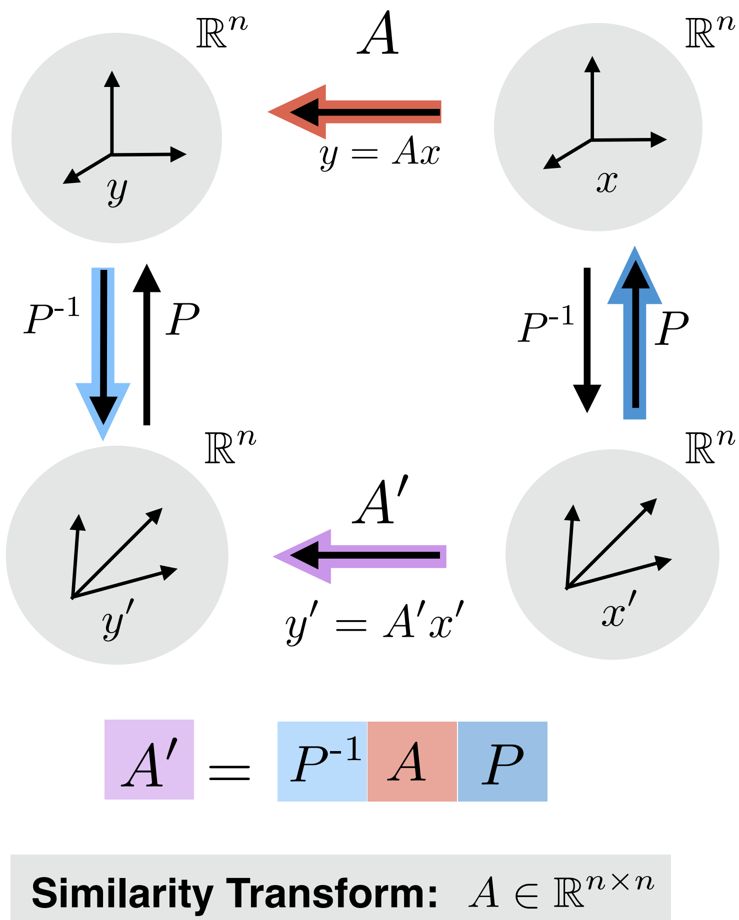


Similarity Tranforms

Now suppose, we have a square matrix $A \in \mathbb{R}^{n \times n}$ that transforms the vector $x \in \mathbb{R}^n$ into a vector $y \in \mathbb{R}^n$, ie. $y = Ax$. We represent both x and y in terms of a new basis given by the columns of $P \in \mathbb{R}^{n \times n}$, ie. $x = Px'$ and $y = Py'$. We want to find a matrix A' that represents the action of A when x and y are expressed in the x' and y' coordinates respectively, ie. we want to find $A' \in \mathbb{R}^{n \times n}$ such that $y' = A'x'$. We can do this by plugging in the relationships between x and z

$$y = Ax \tag{1}$$
$$Py' = APx' \tag{2}$$
$$y' = \underbrace{P^{-1}AP}_{A'}x' \tag{3}$$

We say that A' is related to A by a *similarity transform*, ie. A' represents the transformation of A just with respect to a different coordinate system. The construction of A' is illustrated in the figure below.



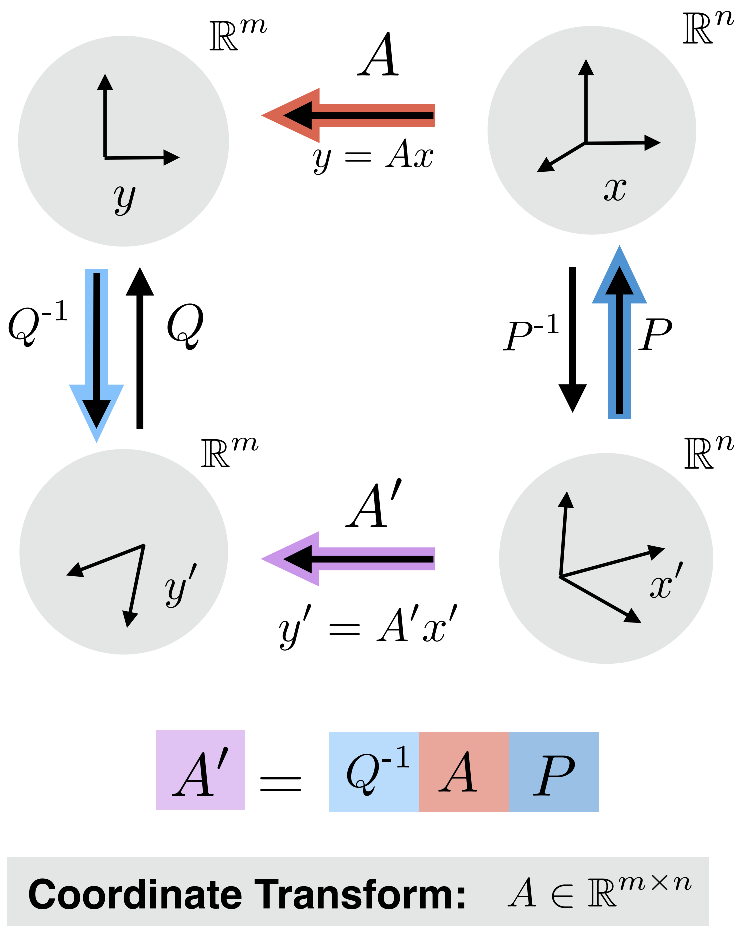
A generalization of this concept is that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are transformed under different coordinate transformations, ie. $x = Px'$ and $y = Qy'$. A is then transformed as

$$y = Ax \tag{4}$$

$$Qy' = APx' \tag{5}$$

$$y' = \underbrace{Q^{-1}AP}_{A'}x' \tag{6}$$

Note that here, it is not necessary that A be square and x and y have the same dimension. This situation is illustrated in the figure below.


$\mathbb{R}^m$
A
$y = Ax$
$\mathbb{R}^n$
y
x
Q^{-1}
Q
P^{-1}
P
$\mathbb{R}^m$
A'
$\mathbb{R}^n$
y'
$y' = A'x'$
x'
$A' = Q^{-1} A P$
Coordinate Transform: $A \in \mathbb{R}^{m \times n}$