AA/EE/ME 510 - Linear System Theory - Fall 2020

Homework 1

<u>**Due Date**</u>: Sunday, Oct 11^{th}, 2020 at 11:59 pm

1. **Inner Products**

   (a) **(PTS: 0-2)** Prove $y^T x = |x||y|\cos\theta$ using the definition of the 2-norm and the law of cosines.

   (b) **(PTS: 0-2)** Prove the parallelogram law:

   $$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$

2. **Fourier Bases**

   The Fourier basis functions are given by

   $$\left\{\sqrt{2}\sin\left(2\pi n t\right) \;\middle|\; n \in \mathbb{N}\right\} \cup \left\{\sqrt{2}\cos\left(2\pi n t\right) \;\middle|\; n \in \mathbb{N}\right\} \cup \left\{1\right\} \tag{1}$$

   where $\mathbb{N}$ is the set of natural numbers. Show that these Fourier basis functions are *orthonormal*, ie. they are orthogonal to each other **(PTS: 0-2)** and they each have magnitude 1 **(PTS: 0-2)**. (Take the limits of integration as $[0, 1]$.)

3. **Projections**

   (a) **(PTS: 0-2)** Compute the projection of $x = [1, 2, 3]^T$ onto $y = [1, 1, -2]^T$.

   (b) **(PTS: 0-2)** Compute the projection of $x = [1, 2, 3]^T$ onto the range of

   $$Y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. **Block Matrix Computations**

   Multiply the following block matrices together. In each case give the required dimensions of the sub-blocks of B. If the dimensions are not determined by the shapes of A, then pick a dimension that works.

   (a) **(PTS: 0-2)**

   $$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1K} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NK} \end{bmatrix}, \qquad AB = ? \tag{2}$$

   where $A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{1N} \in \mathbb{R}^{m_1 \times n_N}$, $A_{M1} \in \mathbb{R}^{m_M \times n_1}$, and $A_{MN} \in \mathbb{R}^{m_M \times n_N}$.

(b) **(PTS: 0-2)**

$$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad AB =? \tag{3}$$

where $A_1 \in \mathbb{R}^{1\times n}$ and $A_m \in \mathbb{R}^{1\times n}$.

(c) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad AB =? \tag{4}$$

where $A_1 \in \mathbb{R}^{m\times 1}$ and $A_n \in \mathbb{R}^{m\times 1}$.

(d) **(PTS: 0-2)**

$$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad D \in \mathbb{R}^{n\times n}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad ADB =? \tag{5}$$

where $A_1 \in \mathbb{R}^{1\times n}$, $A_m \in \mathbb{R}^{1\times n}$.

(e) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad ADB =? \tag{6}$$

where $A_1 \in \mathbb{R}^{m\times 1}$, $A_n \in \mathbb{R}^{m\times 1}$, $d_{ij} \in \mathbb{R}$.

(f) **(PTS: 0-2)**

$$A \in \mathbb{R}^{m\times n}, \quad \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}, \qquad AB =? \tag{7}$$

(g) **(PTS: 0-2)**

$$A = \begin{bmatrix} -A_1- \\ \vdots \\ -A_m- \end{bmatrix}, \quad B, \qquad AB =? \tag{8}$$

where $A_1, A_m \in \mathbb{R}^{1\times n}$.

5. **Norms**

(a) **(PTS: 0-2)** Compute the p-norm of the vector $x = \begin{bmatrix} -1 & 2 & 3 & -2 \end{bmatrix}^T$ for $p = 1, 2, 10, 100, \infty$.

(b) **(PTS: 0-2)** Show that the 'p-norm' for $p = \frac{1}{2}$ does not satisfy the triangle inequality, ie. find examples of $x, y \in \mathbb{R}^2$ such that $|x+y|_{\frac{1}{2}} \not\leq |x|_{\frac{1}{2}} + |y|_{\frac{1}{2}}$.

6. **Vector Derivatives NOTE: DUE WITH HOMEWORK 2**

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. Compute $\frac{\partial f}{\partial x}$ for the following functions:

(a) **(PTS: 0-2)**

$$f(x) = x_1^4 + 3x_1x_2^2 + e^{x_2} + \frac{1}{x_1x_2}$$

(b) **(PTS: 0-2)**

$$f(x) = \begin{bmatrix} \beta x_1 + \alpha x_2 \\ \beta(x_1 + x_2) \\ \alpha^2 x_1 + \beta x_2 \\ \beta x_1 + \frac{1}{\alpha} x_2 \end{bmatrix}$$

for $\alpha, \beta \in \mathbb{R}$

(c) **(PTS: 0-2)**

$$f(x) = \begin{bmatrix} e^{x^T Q x} \\ (x^T Q x)^{-1} \end{bmatrix}$$

for some $Q = Q^T \in \mathbb{R}^{2\times 2}$.