AA/ME/EE 510 - Linear Systems Theory - Autumn 2020

Homework 2

<u>**Due Date**</u>: Sunday, Oct 18^{th}, 2020 at 11:59pm

1. **Vector Derivatives**

   Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. Compute $\frac{\partial f}{\partial x}$ for the following functions:

   (a) **(PTS: 0-2)**

   $$f(x) = x_1^4 + 3x_1x_2^2 + e^{x_2} + \frac{1}{x_1x_2}$$

   (b) **(PTS: 0-2)**

   $$f(x) = \begin{bmatrix} \beta x_1 + \alpha x_2 \\ \beta(x_1 + x_2) \\ \alpha^2 x_1 + \beta x_2 \\ \beta x_1 + \frac{1}{\alpha} x_2 \end{bmatrix}$$

   for $\alpha, \beta \in \mathbb{R}$

   (c) **(PTS: 0-2)**

   $$f(x) = \begin{bmatrix} e^{x^TQx} \\ (x^TQx)^{-1} \end{bmatrix}$$

   for some $Q = Q^T \in \mathbb{R}^{2\times 2}$.

2. **Projections**

   (PTS: 0-2) Consider the matrix $A \in \mathbb{R}^{m\times n}$ (with full column rank and $m > n$) and the two projection matrices $P = A(A^TA)^{-1}A^T$ and $I - P$. Show that $P = P^2$ and $I - P = (I - P)^2$.

3. **Linear Transformations of Sets**

   (a) **Affine Sets:** Consider the affine sets for $x \in \mathbb{R}^2$.

   $$\mathcal{X}_1 = \left\{ x \;\middle|\; x_1 + x_2 = 1,\ x \in \mathbb{R}^2 \right\}, \qquad \mathcal{X}_2 = \left\{ x \;\middle|\; x_1 - x_2 = 1,\ x \in \mathbb{R}^2 \right\},$$

   Draw the set of points Ax for $x \in \mathcal{X}_1$ and $x \in \mathcal{X}_2$ for

   (PTS: 0-2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **(PTS: 0-2)** $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, **(PTS: 0-2)** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(b) **Unit Balls:** Consider the unit-balls defined by the 1-norm, the 2-norm, and the ∞-norm.

$$\mathcal{X}_1 = \left\{x \;\middle|\; |x|_1 \leq 1,\; x \in \mathbb{R}^2\right\}, \qquad \mathcal{X}_2 = \left\{x \;\middle|\; |x|_2 \leq 1,\; x \in \mathbb{R}^2\right\}, \qquad \mathcal{X}_\infty = \left\{x \;\middle|\; |x|_\infty \leq 1,\; x \in \mathbb{R}^2\right\}$$

Draw the set of points Ax for $x \in \mathcal{X}_1$, $x \in \mathcal{X}_2$, and $x \in \mathcal{X}_\infty$ for

(PTS: 0-2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **(PTS: 0-2)** $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, **(PTS: 0-2)** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) **Convex Hulls:** Consider the simplicies in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ respectively

$$\Delta_2 = \left\{x \;\middle|\; \mathbf{1}^T x = 1,\; x \geq 0,\; x \in \mathbb{R}^2\right\},$$
$$\Delta_3 = \left\{x \;\middle|\; \mathbf{1}^T x = 1,\; x \geq 0,\; x \in \mathbb{R}^3\right\},$$
$$\Delta_4 = \left\{x \;\middle|\; \mathbf{1}^T x = 1,\; x \geq 0,\; x \in \mathbb{R}^4\right\}$$

where $\mathbf{1}$ is the vector of all ones of the appropriate dimension and $\geq$ is an element-wise inequality.

Draw the set of points Ax for

(PTS: 0-2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x \in \Delta_2$

(PTS: 0-2) $A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $x \in \Delta_3$

(PTS: 0-2) $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$, $x \in \Delta_4$

4. **Roots of Unity:**

(a) (**PTS: 0-2**) List the complex solutions to the equation $z^4 = 1$, $z \in \mathbb{C}$ (the 4-th roots of unity).

(b) (**PTS: 0-2**) For each 4-th root of unity given above, plot z^t for $t = 0, 1, 2, 3, 4$ in the complex plane.

(c) (**PTS: 0-2**) Consider the complex signals $f_k(t) = e^{\frac{2\pi i k t}{8}}$ for $k = 0, 1, 2, \ldots, 7$. Plot $\mathbf{Re}(f_k(t))$ for $t = 0, 1, 2, \ldots, 7$ for $k = 0, 1, 2, \ldots, 7$.