AA/ME/EE 510 - Linear Systems Theory - Fall 2020

Homework 3

Due Date: Sunday, Oct 25^{th}, 2020 at 11:59pm

1. **Truth Tables**

   For each logical statement, cross out the boxes that are impossible given the statement in the upper-left corner.

   (a) **(PTS: 0-2)**

p $\wedge q$	q	$\neg q$
p		
$\neg p$		

p $\vee q$	q	$\neg q$
p		
$\neg p$		

$(p \wedge \neg q) \vee (\neg p \wedge q)$	q	$\neg q$
p		
$\neg p$		

   (b) **(PTS: 0-2)**

p $\Rightarrow q$	q	$\neg q$
p		
$\neg p$		

p $\Leftarrow q$	q	$\neg q$
p		
$\neg p$		

p $\iff q$	q	$\neg q$
p		
$\neg p$		

2. **Coordinates**

   Let y be the coordinates of a vector with respect to the standard basis in $\mathbb{R}^2$. In each case below consider a different basis for $\mathbb{R}^2$ given by the columns of the matrix T. Compute the coordinates of the vector y with respect to the new basis 1) by graphically drawing the columns of T and y as vectors and 2) by inverting the matrix T, ie. by solving $y = Tx$.

   (a) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (b) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

   (c) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

(d) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

3. **Block Matrix Inversion**

Consider the block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

(a) **(PTS: 0-2)** Show that

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

(b) **(PTS: 0-2)** Show that

$$\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & 0 \\ 0 & F^{-1} \end{bmatrix}$$

(c) **(PTS: 0-2)** Show that

$$\begin{bmatrix} I & G \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -G \\ 0 & I \end{bmatrix}$$

(d) **(PTS: 0-2)** Show that

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

Note: you can do this either by using the first three parts or by showing directly that $M^{-1}M = I$.

4. **Woodbury Matrix Identity**

Let $M = A + UCV$ where $M, A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times m}$ and let $n > m$.

(a) **(PTS: 0-2)** What are the dimensions of U and V? Which one is tall and which one was fat?

(b) **(PTS: 0-2)** Show the Woodbury Matrix Identity

$$M^{-1} = (A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}.$$

5. **Steinitz Exchange Lemma**

For a vector space $\mathcal{V}$, let the columns of V be a linearly independent set of m vectors and let the columns of W span all of $\mathcal{V}$.

$$V = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_m \\ | & & | \end{bmatrix}, \qquad W = \begin{bmatrix} | & & | \\ W_1 & \cdots & W_n \\ | & & | \end{bmatrix}$$

Show that for $k \leq m$, you can always select $n-k$ cols of W so that the columns of

$$V^k = \begin{bmatrix} | & & | & | & & | \\ V_1 & \cdots & V_k & W_{k+1} & \cdots & W_n \\ | & & | & | & & | \end{bmatrix},$$

span all of $\mathcal{V}$. Use an inductive argument by following these steps.

(a) **(PTS: 0-2)** Show that the columns of

$$V^0 = W = \begin{bmatrix} | & & | \\ W_1 & \cdots & W_n \\ | & & | \end{bmatrix}$$

span $\mathcal{V}$.

(b) **(PTS: 0-2)** Show that if the columns of

$$V^{k-1} = \begin{bmatrix} | & & | & | & & | \\ V_1 & \cdots & V_{k-1} & W_k & \cdots & W_n \\ | & & | & | & & | \end{bmatrix},$$

span $\mathcal{V}$, then the columns of

$$V^k = \begin{bmatrix} | & & | & | & & | \\ V_1 & \cdots & V_k & W_{k+1} & \cdots & W_n \\ | & & | & | & & | \end{bmatrix},$$

span $\mathcal{V}$. (Note that if you need to, you can reorder the columns of W at any point.)

Note: Setting $k = m$, shows that $m \leq n$ and that a set of spanning vectors can always be used to augment a set of linearly independent vectors to create a basis for a finite dimensional vector space $\mathcal{V}$. This is called *completing a basis*.