AE 510 - Linear Systems Theory - Winter 2020

Homework 4

<u>**Due Date**</u>: Sunday, Nov 1^{st}, 2020 at 11:59pm

1. **Elementary Matrices and Matrix Inverses**

   Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   (a) **(PTS: 0-2)** Compute a sequence of elementary matrices that could be used to row-reduce A to the identity.

   (b) **(PTS: 0-2)** Use this sequence of elementary matrices to compute A^{-1}.

2. **Similarity Transformations**

   Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   and the equation $y = Ax$ for $x, y \in \mathbb{R}^2$. For each coordinate transformation $T \in \mathbb{R}^{2\times 2}$ shown below, compute the matrix A' such that $y' = A'x'$ when $x = Tx'$ and $y = Ty'$.

   (PTS: 0-2) $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, **(PTS: 0-2)** $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

3. **Rotation Matrices**

   Recall that a rotation matrix is a matrix $R \in \mathbb{R}^{n\times n}$ that satisfies $R^T R = I$ and $\det(R) = 1$.

   (a) **(PTS: 0-2)** Consider $R \in \mathbb{R}^{n\times n}$. Show that if R is a rotation matrix, then its inverse is also a rotation matrix.

   (b) **(PTS: 0-2)** Consider $R_1, R_2 \in \mathbb{R}^{n\times n}$ and $R = R_1 R_2$. Prove that if R_1 and R_2 are rotation matrices, then R is also a rotation matrix.

4. **Finding a Nullspace Basis**

   (a) **Basis Derivation**

   Consider a fat matrix $A \in \mathbb{R}^{m\times n}$ $(m < n)$ that is partitioned as $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ with $A_1 \in \mathbb{R}^{m\times m}$ invertible. Show that the columns of $B \in \mathbb{R}^{n\times n-m}$

$$B = \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix}$$

   form a basis for the nullspace of A, $\mathcal{N}(A)$ by performing the following two steps.

i. **(PTS: 0-2)** Show that any vector $v \in \mathcal{N}(A)$ can be written as $v = Bw$ for some $w \in \mathbb{R}^{n-m}$, ie. v is linear combination of the columns of B (the columns of B span the nullspace).

ii. **(PTS: 0-2)** Show that the columns of B are linearly independent.

(b) **Computation**

For the following matrices explicitly compute a basis for their nullspaces BY HAND, ie. do not use computational software.

i. **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

ii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix}$$

iii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{bmatrix}$$

5. **Matrix Rank**

The column rank of a matrix is the number of linearly independent columns. The row rank of a matrix is the number of linearly independent row.

(a) **(PTS: 0-2)** Show that the row rank is less than or equal to the column rank.

(b) **(PTS: 0-2)** Show that the col rank is less than or equal to the row rank.

6. **Range and Nullspace**

Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent the range and nullspace of A (and similarly let $\mathcal{R}(A^T)$ and $\mathcal{N}(A^T)$ be the range and nullspace of A^T).

(a) **(PTS: 0-2)** Suppose $y \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$. Show that $x \perp y$, ie. $x^T y = 0$.

(b) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{5\times 10}$. Suppose A has only 3 linearly independent columns (the other 7 are linearly dependent on the first 3). What is the dimension of $\mathcal{R}(A)$? What is the dimension of $\mathcal{N}(A^T)$?

(c) **(PTS: 0-2)** What is the dimension of $\mathcal{N}(A)$? What is the dimension of $\mathcal{R}(A^T)$?

7. **Least Squares and Minimum Norm Solutions**

(a) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{m\times n}$ where $m > n$ (A is ”tall”) and A has full-column rank (the columns are linear independent). Show that the least squares solution $x = (A^\intercal A)^{-1} A^\intercal y$, minimizes $|y - Ax|^2$, ie. makes Ax as close as possible to y.

(b) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{m\times n}$ where $m < n$ (A is ”fat”) and A has full-row rank (the rows are linear independent). Let $x = A^\intercal (AA^\intercal)^{-1} y$ and $z \in \mathbb{R}^n$ be any vector such that $y = Az$. Show that $|x| \leq |z|$.