# AA/ME/EE 510 - Linear Systems Theory - Fall 2020

# Homework 5

**Due Date**: Sunday, Nov $8^{th}$, 2020 at 11:59pm

1. **Lagrange Multipliers - Nonlinear Programming**

   Consider the following constrained optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \tfrac{1}{2}(x_1^2 + x_2^2) \\ \text{s.t.} \quad & x_2 = -\tfrac{1}{2}(x_1 - 1)^2 \end{aligned}$$

   (a) **(PTS: 0-2)** Draw the level sets of the objective function and the constrained set on a plot in $\mathbb{R}^2$.

   (b) **(PTS: 0-2)** Write out the Lagrangian $\mathcal{L}(x, \lambda)$ with a dual variable $\lambda \in \mathbb{R}$ and the optimality conditions.

   (c) **(PTS: 0-2)** Solve for the optimal primal variable $x \in \mathbb{R}^2$ and the optimal dual variable $\lambda \in \mathbb{R}$ by solving the optimality conditions.

   (d) **(PTS: 0-2)** Solve the problem again with new objective function $5(x_1^2 + x_2^2)$ and replace $\lambda$ with $-\lambda$ in your Lagrangian. How does the optimal $x$ change? How does the optimal $\lambda$ change?

2. **Lagrange Multipliers - Convex Quadratic Programming**

   Consider the following constrainted optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \tfrac{1}{2}x^T Q x - c^T x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

   where $Q \in \mathbb{R}^{n \times n}$, $Q = Q^T \succ 0$ (symmetric, positive definite), $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ $(m < n)$ and $b \in \mathbb{R}^m$.

   (a) **(PTS: 0-2)** Write out the Lagrangian $\mathcal{L}(x, \lambda)$ and the optimality conditions with dual variables $\lambda \in \mathbb{R}^m$.

   (b) **(PTS: 0-2)** Transform the optimality conditions into a system of $m+n$ linear equations for the variables $\lambda$ and $x$.

   (c) **(PTS: 0-2)** Show that

$$\begin{bmatrix} A^T & Q \\ 0 & A \end{bmatrix}^{-1} = \begin{bmatrix} (AQ^{-1}A^T)^{-1}AQ^{-1} & -(AQ^{-1}A^T)^{-1} \\ Q^{-1} - Q^{-1}A^T(AQ^{-1}A^T)^{-1}AQ^{-1} & Q^{-1}A^T(AQ^{-1}A^T)^{-1} \end{bmatrix}$$

   (d) **(PTS: 0-2)** Solve for the optimal $x$ and $\lambda$ in terms of $Q, c, A, b$.

3. **Grammian Rank**

   **(PTS: 0-2)** Show that $A^T A$ has the same rank as $A \in \mathbb{R}^{m \times n}$. Hint: use the rank-nullity theorem and show that both matrices have identical nullspaces.

4. **Basis for Domain from Nullspace of $A$ and Range of $A^T$**

   Consider $A \in \mathbb{R}^{m \times n}$ with $m < n$ and full row rank and a matrix $N \in \mathbb{R}^{n \times n-m}$ with full column rank whose columns span the nullspace of $A$. Suppose we write a vector $x \in \mathbb{R}^n$ as a linear combination of the rows of $A$ and the columns of $N$, ie.

$$x = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

   for $x_1' \in \mathbb{R}^m$ and $x_2' \in \mathbb{R}^{n-m}$

   (a) **(PTS: 0-2)** Symbollically compute $\begin{bmatrix} A^T & N \end{bmatrix}^{-1}$.

   Hint: Start by checking if $\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} A^T & N \end{bmatrix}^T$...

   (b) **(PTS: 0-2)** Solve for $x_1'$ and $x_2'$ given $A$,$N$, and $x$.

5. **Sylvester's Inequality**

   Prove Sylvester's inequality.

$$\mathrm{rk}(A) + \mathrm{rk}(B) \leq \mathrm{rk}(AB) + n$$

   for matrices $A, B \in \mathbb{R}^{n \times n}$

   (a) **(PTS: 0-2)** Show that this inequality can be rewritten as

$$n - \mathrm{rk}(A) + n - \mathrm{rk}(B) \geq n - \mathrm{rk}(AB)$$

   and interpret this new form in terms of the dimensions of the nullspaces of $A$,$B$, and $AB$.

   (b) **(PTS: 0-2)** Prove Sylvester's inequality.

6. **Fundamental Theorem of Linear Algebra Pictures**

   For each of the following matrices draw a picture of the domain (either $\mathbb{R}^2$ or $\mathbb{R}^3$) labeling $\mathcal{R}(A^T)$ and $\mathcal{N}(A)$ and a picture of the co-domain (either $\mathbb{R}^2$ or $\mathbb{R}^3$) labeling the $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$.

   **(PTS: 0-2)** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   **(PTS: 0-2)** $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

   **(PTS: 0-2)** $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$

   **(PTS: 0-2)** $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$