AE 510 - Linear Systems Theory - Winter 2020

Homework 6

<u>**Due Date**</u>: Wednesday, Nov 25^{th}, 2020 at 11:59pm

1. **Spectral Mapping Theorem**

   Consider a diagonalizable matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a polynomial function $f : R^{n\times n} \to R^{n\times n}$.

   (a) **(PTS: 0-2)** Show that the eigenvectors (left and right) of $f(A)$ are the same as the eigenvectors of A.

   (b) **(PTS: 0-2)** Show that the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

2. **Cayley-Hamilton Theorem**

   (a) **(PTS: 0-2)** The eigenvalues of a matrix A are roots of its characteristic polynomial, $\chi(\lambda) = \det(\lambda I - A)$, ie. $\det(\lambda_i I - A) = 0$ if λ_i is an eigenvalue of A. Show that $\chi(A) = \mathbf{0}$ (where $\mathbf{0}$ is a matrix of zeros). (Hint: use the spectral mapping theorem).

   (b) **(PTS: 0-2)** . Suppose that $\chi(\lambda) = \det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda - 1$. Use Cayley-Hamilton to write an expression for A^6 in terms of A^2, A, I. Note that when you plug the matrix A into $\chi(\cdot)$ you replace each constant with that constant times the identity matrix, ie. $\chi(A) = A^3 - 2A^2 + A - I$.

3. **Computing Eigenvalues and Diagonalization**

   Compute eigenvalues and right eigenvectors for each of the following matrices. Write out a diagonalization for each matrix. If the matrix has complex eigenvalues, then write it in both of these forms.

$$\begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}(u - vi) & \frac{1}{\sqrt{2}}(u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} a + bi & 0 \\ 0 & a - bi \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}}(w^T + y^T i)- \\ -\frac{1}{\sqrt{2}}(w^T - y^T i)- \end{bmatrix} = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} -w^T- \\ -y^T- \end{bmatrix}$$

   (a) **(PTS: 0-2)** Eigenvalues, Eigenvectors, **(PTS: 0-2)** Diagonal form, Complex form?

$$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

   (b) **(PTS: 0-2)** Eigenvalues, Eigenvectors, **(PTS: 0-2)** Diagonal form, Complex form?

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

   (c) **(PTS: 0-2)** Eigenvalues, Eigenvectors, **(PTS: 0-2)** Diagonal form, Complex form?

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

4. **Rotation Matrices and Complex Eigenvectors**

Consider the two rotation matrices

$$R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

(a) **(PTS: 0-2)** Show that R_1 and R_2 commute, ie. $R_1R_2 = R_2R_1$ (Note that most matrices do not commute. 2×2 rotation matrices are an exception.)

(b) **(PTS: 0-2)** Compute the inverse of R_1.

(c) **(PTS: 0-2)** Give a physical interpretation of R_1R_2 and R_1^{-1} related to the angles θ_1 and θ_2.

(d) **(PTS: 0-2)** Consider a 2×2 real matrix A that can be diagonalized as

$$A = \tfrac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ (u - vi) & (u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{bmatrix} \left(\begin{bmatrix} | & | \\ (u - vi) & (u + vi) \\ | & | \end{bmatrix} \frac{1}{\sqrt{2}} \right)^{-1}$$

where $r \in R_+$ and $u, v \in R^2$. Show that another valid diagonalization for A is

$$A = \tfrac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ (u' - v'i) & (u' + v'i) \\ | & | \end{bmatrix} \begin{bmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{bmatrix} \left(\begin{bmatrix} | & | \\ (u' - v'i) & (u' + v'i) \\ | & | \end{bmatrix} \frac{1}{\sqrt{2}} \right)^{-1}$$

where $u' = \cos(\phi)u + \sin(\phi)v$ and $v' = -\sin(\phi)u + \cos(\phi)v$ for any angle ϕ.

5. **Orthogonal Eigenvectors**

Suppose $p_1, p_2 \in R^2$ are linearly independent right eigenvectors of $A \in R^{2\times2}$ with eigenvalues $\lambda_1, \lambda_2 \in R$ such that $\lambda_1 \neq \lambda_2$. Suppose that

$$p_1^T p_2 = 0, \qquad |p_1| = 1, \qquad |p_2| = 2$$

(a) **(PTS: 0-2)** Write an expression for a 2×2 matrix whose rows are the left-eigenvectors of A

(b) **(PTS: 0-2)** Write an expression for a similarity transform that transforms A into a diagonal matrix.

6. **Traces and Determinants**

Assume that $A \in R^{n\times n}$ is diagonalizable and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. Use the properties of traces and determinants to show that

(a) **(PTS: 0-2)**: $\quad \text{Tr}(A) = \sum_i \lambda_i$

(b) **(PTS: 0-2)**: $\quad \det(A) = \prod_i \lambda_i$

7. **Similar Eigenvalues**

(a) **(PTS: 0-2)** Let $A \in \mathbb{R}^{n\times n}$ and let $T \in \mathbb{R}^{n\times n}$ be any non-singular matrix. Show that the eigenvalues of A are the same as those of $T^{-1}AT$. What are the eigenvectors of $T^{-1}AT$?

(b) **(PTS: 0-2)** Let $A, B \in \mathbb{R}^{n\times n}$ be invertible matrices. Show that the eigenvalues of AB are the same as those of BA.