AE 510 - Linear Systems Theory - Winter 2020

Homework 7

<u>Due Date</u>: Monday, Dec 7^{th}, 2020 at 11:59pm

1. **Nilpotent Matrices**

   (PTS:0-2) Consider the matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

   What is N^2? What is N^3? What is N^4? What is the characteristic polynomial of N? What are the eigenvalues of N?

2. **Jordan Form**

   Consider a matrix $A \in \mathbb{R}^{3\times 3}$ with the form

$$A = \begin{bmatrix} | & | & | \\ v_1 & w_1 & v_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & a & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} | & | & | \\ v_1 & w_1 & v_2 \\ | & | & | \end{bmatrix}^{-1}$$

   - **(PTS: 0-2)** What is the characteristic polynomial of A? Write an expression for A^{-1}.
   - **(PTS: 0-2)** Write an expression for the Jordan form of $A = PJP^{-1}$, ie. find P and J. What is the 1st order generalized eigenvector associated with λ_1? What is the 2nd order generalized eigenvector associated with λ_1? What happens when to the Jordan form when $a \to \infty$? What about when $a \to 0$?

3. **Positive Definite Matrices and Congruent Transformations**

   - **(PTS: 0-2)** Show that if $K \in \mathbb{R}^{n\times n}$ is skew-symmetric, i.e. $K^T = -K$, then $\mathbf{x}^T K \mathbf{x} = 0$ for all $x \in \mathbb{R}^n$.
   - **(PTS: 0-2)** Consider the quadratic expression $\mathbf{x}^T A \mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Show that for all A, there exists a matrix $B \in \mathbb{R}^{n\times n}$ with $B^T = B$ such that

$$\mathbf{x}^T B \mathbf{x} = \mathbf{x}^T A \mathbf{x}.$$

   - **(PTS: 0-2)** Show that if $A \in \mathbb{R}^{n\times n}$ is positive definite then $Q = P^T A P$ is positive semi-definite for any $P \in \mathbb{R}^{n\times m}$.

4. **Eigenvectors of Symmetric and Skew Symmetric Matrices**

- **(PTS: 0-2)** Show that the eigenvalues of symmetric matrices are always real. Show that the eigenvectors of symmetric matrices are orthogonal.
- **(PTS: 0-2)** Show that the eigenvalues of skew-symmetric matrices are always imaginary. Show that the eigenvectors of skew-symmetric matrices are orthogonal.

5. **Continuous vs. Discrete Time Stability**

(PTS: 0-2) Let $\rho(M)$ represent the set of eigenvalues or *spectrum* of M. Show that if $\text{Re}(\lambda) < 0$ for all $\lambda \in \rho(A)$, then $|\mu| < 1$ for all $\mu \in \rho(e^{A\Delta t})$.

6. **Vector Fields and Stability**

For each of the following matrices,

$$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

consider the system differential equation

$$\dot{x} = Ax$$

- **(PTS: 0-2 (for each))** What are the eigenvalues of e^{At}? Is the system stable? Sketch the vector field for each system labeling the eigenvectors and show a sample trajectory.

7. **Singular Value Decomposition**

- **(PTS:0-4)** Show that any matrix $A \in \mathbb{R}^{m\times n}$ can be written in the form

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$$

for $U \in \mathbb{C}^{m\times m}$, $U^*U = I$ and for $V \in \mathbb{C}^{n\times n}$, $V^*V = I$, $\Sigma \in \mathbb{R}^{k\times k}$ diagonal and $\Sigma \succ 0$ where k is the rank of A. Let

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \qquad V = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}$$

where $U_1 \in \mathbb{C}^{m\times k}$, $U_2 \in \mathbb{C}^{m\times (m-k)}$, $V_1 \in \mathbb{C}^{n\times k}$, $V_2 \in \mathbb{C}^{n\times (n-k)}$. What spaces are spanned by the columns of U_1, U_2, V_1, and V_2?

- **(PTS:0-2)** Write expressions for the following matrices in terms of $U_1, U_2, V_1, V_2, \Sigma$.

$$A^T = ?, \quad , A^{-1} = ?, \quad A^{-T} = ?$$

$$(A^TA)^{\frac{1}{2}} = ?, \quad (AA^T)^{\frac{1}{2}} = ?, \quad (A^TA)^{-1}A^T = ?, \quad A^T(AA^T)^{-1} = ?$$