APPROACH:

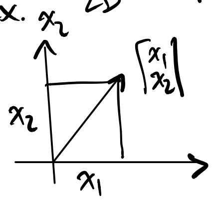
- spatial intuition
- tricks
- rigor of proofs

Vector Space

$\boxed{x \in \mathbb{R}^n}$ $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

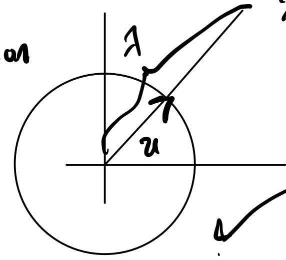
"magnitude & direction"

Ex. 2D

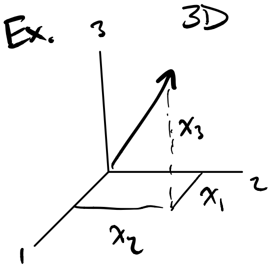


Cartesian representation

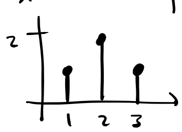
$x = \lambda u$



Ex. 3D



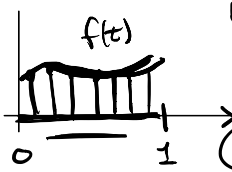
$x \in \mathbb{R}^3$ $\quad x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$



$x = x_i$ — value, index

x_i

Ex. $f(t) \in C([0,1])$ ← vector space



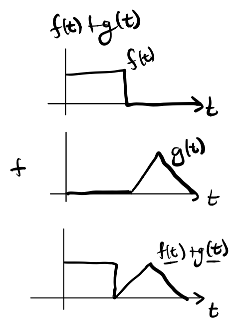
t → index continuous

can also be thought of a "vector infinite dim vector"

$f(t)$ — value, index

$f(\cdot)$ $\quad f(t)$

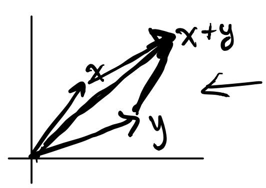
$f(t) + g(t)$



Adding vectors:

"add elementwise"

$$x + y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$



Multiplying Vectors:

side note: crossproduct

$$\underset{\mathbb{R}^3}{x \times y} \Rightarrow \underbrace{\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}}_{\hat{x}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \hat{x} y = x \times y$$

skew symmetric

← aside

DOT PRODUCT or INNER PRODUCT

Notation $x \cdot y = \langle x, y \rangle = \boxed{x^T y} = \sum_i x_i y_i$

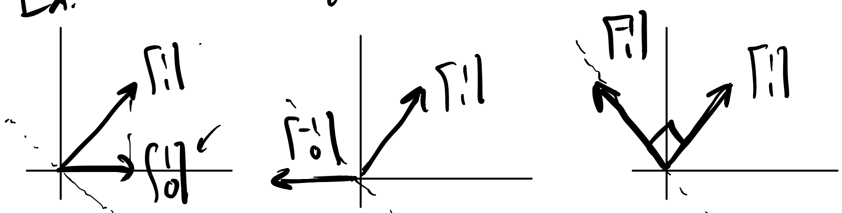
fin. dim — inf. dim — matrix multi version

functional analysis

$\langle x | y \rangle$

quantum mech.

Ex.



$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$ pos same direc.

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1$ neg op direct.

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0$ perpendicular

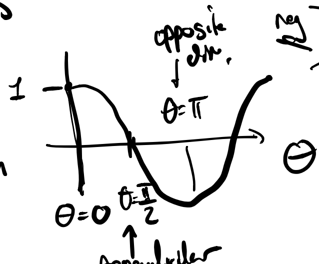
$\boxed{\text{orthogonal}}$

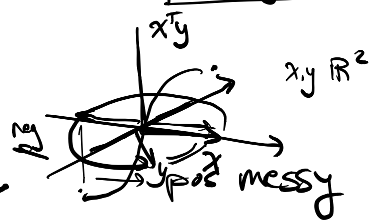
inner product:
- like scalar mult. : "lengths multiply"
- do the directions line up.

$x^T y = |x||y| \cos\theta$

angle in between

prove





$\theta = 0 \quad x^T y = |x||y|$

$\theta = \frac{\pi}{2} \quad x^T y = 0$

$\theta = \pi \quad x^T y = -|x||y|$

Norms (Length)

Standard def of length

$$2\cdot\text{norm} = \sqrt{\sum_i x_i^2} = (x^T x)^{1/2}$$

norm "induced by an inner product" ← Hilbert spaces have a metric

$\|x\|_2 = |x|_2 = |x|$ assume 2-norm unless otherwise stated

Tech defn: $|\cdot|$

1. $|x| = 0$ only if $x = 0$
2. $|ax| = |a||x|$
3. $|x+y| \le |x| + |y|$ triangle inequality

other norms:

1-norm $\|x\|_1 = |x|_1 = \left(\sum_i |x_i|\right)$

2-norm $\|x\|_2 = |x|_2 = \left(\sum_i |x_i|^2\right)^{1/2}$

p-norm $\|x\|_p = |x|_p = \left(\sum_i |x_i|^p\right)^{1/p}$, $1 \le p < \infty$

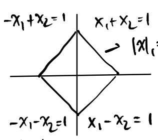
∞-norm $\|x\|_\infty = |x|_\infty = \left(\sum_i |x_i|^\infty\right)^{1/\infty} = \max_i |x_i|$

unit ball: $\{x \mid \|x\|_p = 1\}$ = unit ball of p norm

unit vector, length 1

$$x = \frac{x}{|x|}$$

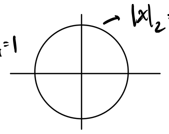
$$x = |x|\left(\frac{x}{|x|}\right)$$

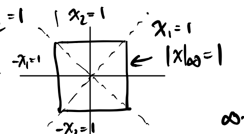


$|x_1| + |x_2| = 1$

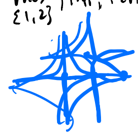
$x_1 + x_2 = 1$

unit balls need to be convex

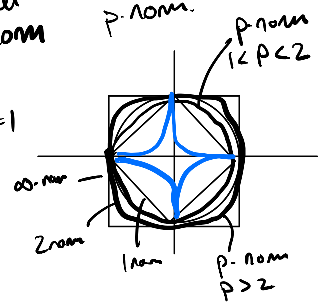




$\max_{\{1,2\}} \{|x_1|, |x_2|\}$



p-norm.



Inner products of functions

$f(t)$ $g(t)$ $t \in [0,1]$

$$\langle f, g\rangle = \int f(t)g(t)\,dt$$

$f(x)$ $g(x)$ $x \in X \subset \mathbb{R}^2$

$$\langle f, g\rangle = \int_X f(x)g(x)\,dx$$

make sure this is defined

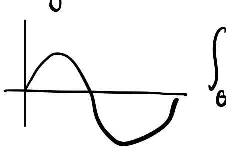
orthogonal functions

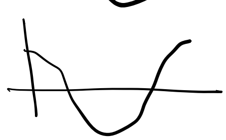
function space L^2

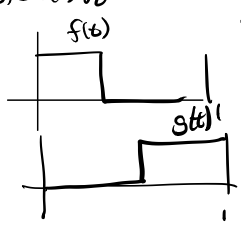
$$\int_0^{2\pi} \sin(t)\cos(t)\,dt$$

$$\int f(t)^2\,dt < \infty$$

$$\int_0^1 f(t)g(t)\,dt = 0$$







Mathematical Spaces (topology)

some notion of objects being near ea. other

Ex $\mathbb{R}^n$

$\mathbb{C}^n$

→ $C[0,1]$

→ $C^1[0,1]$

→ $C^2[0,1]$

$C^\infty[0,1]$

L^2

- Hilbert space (inner product)
- metric space

outer product: inner product $x, y \in \mathbb{R}^n$

$xy^T \in \mathbb{R}^{n \times n}$ $x^T y \in \mathbb{R}$

$z \in \mathbb{R}^n$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [y_1 \cdots y_n] = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{bmatrix} \qquad = \underline{xy^T z}$$

dyad

rank 1

dim 1 $\quad x \underbrace{y^T z}_{\mathbb{R}}$

$$\alpha x = [x] (y^T z)$$

Matrix multiplication:

$A \in \mathbb{R}^{m \times n}$

$B \in \mathbb{R}^{n \times p}$

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots \\ \vdots & \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

2 other forms:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} - \bar{a}_1^T - \\ \vdots \\ - \bar{a}_m^T - \end{bmatrix} \overset{①}{=} \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} ②$$

"stack of rows" "row of columns"

① : $$Ax = \begin{bmatrix} - \bar{a}_1^T - \\ \vdots \\ - \bar{a}_m^T - \end{bmatrix} x = \begin{bmatrix} \bar{a}_1^T x \\ \vdots \\ \bar{a}_m^T x \end{bmatrix}$$ ⟸ very important

"take dot product of x w/ ea. row of A"

② $$\boxed{Ax = [A_1 \cdots A_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A_1 x_1 + \cdots + A_n x_n}$$ ⟸ super blisteringly important

"taking dot product of x w/ a row of cols of A"

linear combination of cols of A

Block Matrix multiplication

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots \\ \vdots & \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots \end{bmatrix}$$

$$① \begin{bmatrix} \overset{*}{A_{11}} & \cdots & \overset{**}{A_{1n}} \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} * & B_{11} & \cdots & B_{1p} \\ ** & B_{n1} & & B_{np} \end{bmatrix}^{②} = \begin{bmatrix} ① & \overset{②}{A_{11}B_{11} + \cdots + A_{1n}B_{n1}} & \cdots & (\quad) \\ & \vdots & & \\ & A_{m1}B_{11} + \cdots + A_{mn}B_{n1} & \cdots & (\quad) \end{bmatrix}$$

2 specific cases:

$$AB = \begin{bmatrix} — \bar{a}_1^T — \\ \vdots \\ — \bar{a}_m^T — \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} \overset{\mathbb{R}}{\bar{a}_1^T B_1} & \cdots & \bar{a}_1^T B_p \\ \vdots & & \\ \bar{a}_m^T B_1 & \cdots & \bar{a}_m^T B_p \end{bmatrix}$$

a matrix of scalars

$$AB = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \begin{bmatrix} — \bar{b}_1^T — \\ \vdots \\ — \bar{b}_n^T — \end{bmatrix} = \underbrace{A_1 \bar{b}_1^T}_{\mathbb{R}^{n \times n}} + \cdots + A_n \bar{b}_n^T$$

sum of matrices

dyad expansion

$$AB = A\begin{bmatrix} B_1 & \cdots & B_p \end{bmatrix} = \begin{bmatrix} AB_1 & \cdots & AB_p \end{bmatrix}$$