$$x + y = 1 \Rightarrow 1 + \frac{y}{x} = \frac{1}{x} \quad \boxed{x \neq 0}$$

$$A^{-1}\left[Ax = y\right] \Rightarrow A^{-1}y = x \quad \boxed{A^{-1} \text{ exists}}$$

A full

$$\begin{bmatrix} v^T \\ v_2^T \end{bmatrix} \left[A\right] x = v^T \left[y\right] \Rightarrow \underbrace{v^T A x}_{\text{scalar}} = \underbrace{v^T y}_{\text{scalar}} \quad \text{1 eqn}$$

m eqns.

$$w^T \left[A\right] x = w^T y \qquad \underbrace{w^T A x}_{\text{scalar}} = \underbrace{w^T y}_{\text{scalar}}$$

m eqns $\rightarrow$ 2 equs or 1 eqn etc

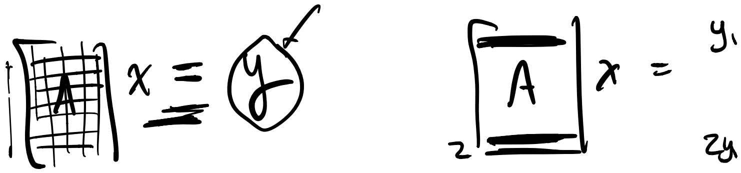
$$v^T\left[Ax = y\right] \Rightarrow 0 = 0$$

$$v^T A = 0 \quad v^T y = 0$$

full col rank

$$(A^T A)^{-1} A^T \left[Ax = y\right] = \boxed{x = (A^T A)^{-1} A^T y}$$

$$M\left[Ax = y\right] \qquad MAx = My$$



not full row rank = extra equations ←
(repeats) $\boxed{\text{fear } y \notin R(A)}$

not full col rank = redundant parameters
two x's give you the same y.

↓ → not full col rank

$\underbrace{(A^TA)^{+}}_{\text{pseudo inverse}} A^T A x =$ $(A^TA)^{+}A^T y = x + \cancel{z}^{\,0}$ $z \in N(A)$

a solution to $Ax = y$.

$x \perp N(A)$

minimum norm soln.

$\boxed{\text{Moore-Penrose pseudo inverse}}$

↳ for any $A \in \mathbb{R}^{m \times n}$

A is tall ⇒ leastsqueres

A is fat ⇒ minimum norm

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^* \qquad A^{+} = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$$

$$Ax = y \qquad A\underline{A^T(AA^T)^{-1}} = I$$

$$\left(A^T \right) \left[- A \right] \left[x \right] = \left[y \right]$$

full row rank

Guess:

$$\underline{x} = \underline{A^T(AA^T)^{-1}}y$$

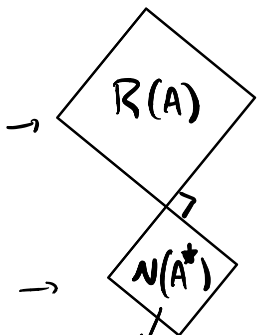
$$\underbrace{A\left(A^T(AA^T)^{-1}\right)}_{I}y = y. \checkmark$$

$$x = A^T(AA^T)^{-1}y + z \qquad z \in N(A)$$

Review: Fundamental Thm of LA

$y \in \mathbb{C}^m$ CODOMAIN $\xleftarrow{\;A \in \mathbb{C}^{m\times n}\;}$ $x \in \mathbb{C}^n$ DOMAIN

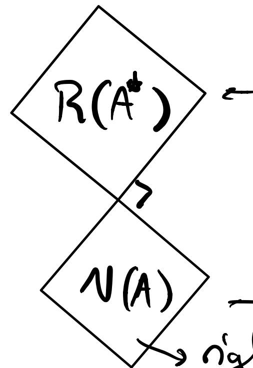
$y = Ax$



left nullspace

$y^* A = 0$

real case: $y^T A = 0$

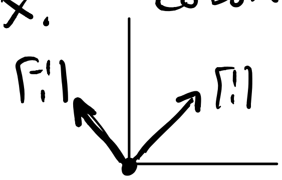


right nullspace

$$\mathbb{C}^n = R(A^*) \oplus N(A)$$

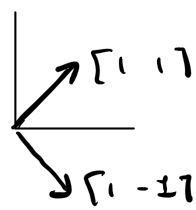
$$\mathbb{C}^m = R(A) \oplus N(A^*)$$

Ex. CODOM $= \mathbb{R}^2$



$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \longleftarrow$$

DOM $= \mathbb{R}^2$

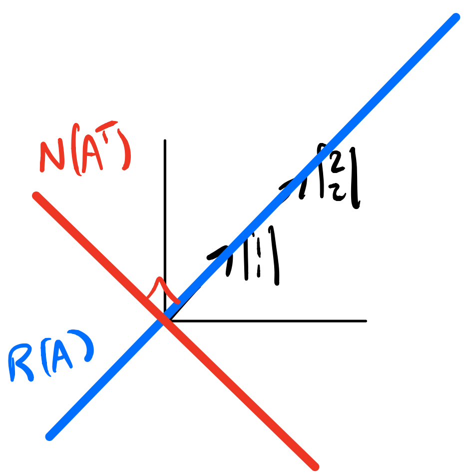


$R(A) = \mathbb{R}^2$

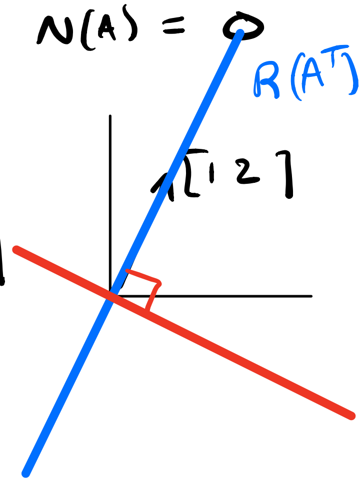
$N(A^T) = 0$

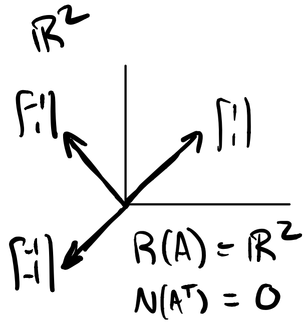
$R(A^T) = \mathbb{R}^2$

$N(A) = 0$



$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

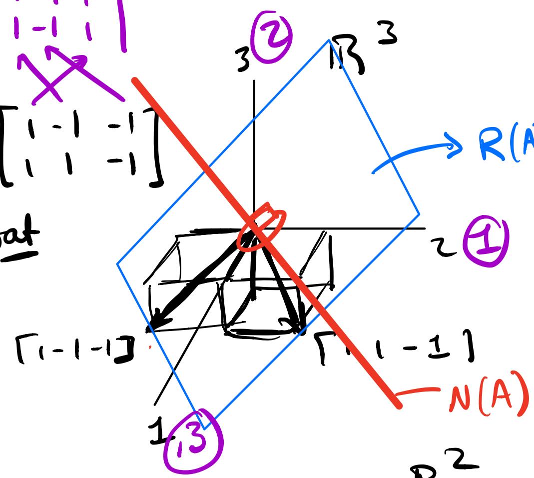


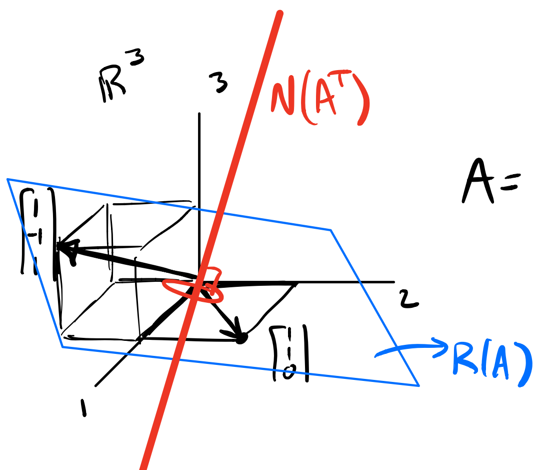


$\begin{bmatrix}-1 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}$

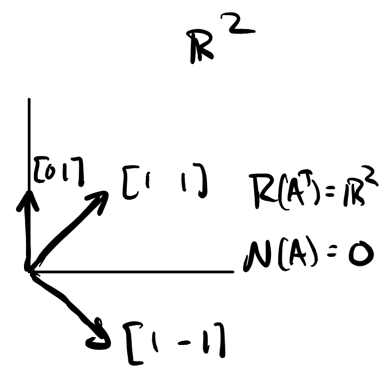
$$A = \begin{bmatrix}1 & -1 & -1\\ 1 & 1 & -1\end{bmatrix}$$

Sol



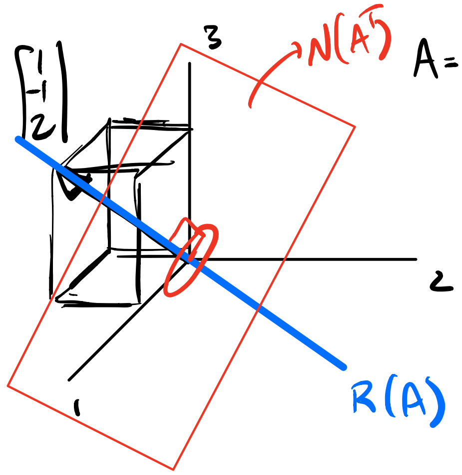


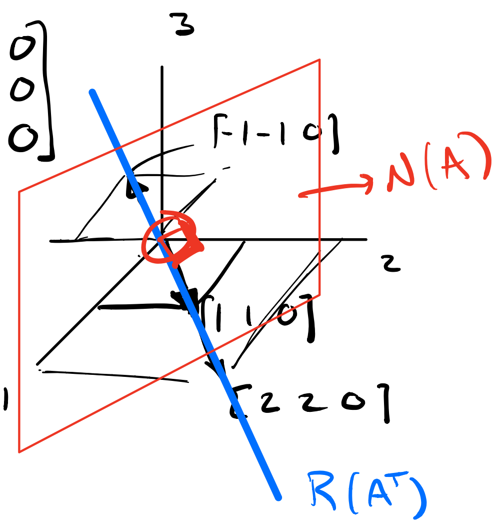
$$A = \begin{bmatrix}1 & 1\\ 1 & -1\\ 0 & 1\end{bmatrix}$$



rank 1

$$A = \begin{bmatrix}1 & 1 & 0\\ -1 & -1 & 0\\ 2 & 2 & 0\end{bmatrix}$$





$R(A^T) \perp N(A)$

$x_1 \in R(A^T)$ $\quad$ $x_2 \in N(A) \rightarrow \boxed{Ax_2 = 0}$

$\exists z \ \boxed{x_1 = A^T z}$

$$x_1^T x_2 = z^T \underbrace{A x_2}_{0} = 0$$

$$M = \begin{array}{c} \\ n \end{array} \begin{array}{c} n \\ \left[\begin{array}{cc} B & C \end{array}\right] \\ \begin{array}{cc} n_1 & n_2 \end{array} \end{array} \qquad \begin{array}{l} M \in \mathbb{R}^{n \times n} \ \text{invertible} \\ B \in \mathbb{R}^{n \times n_1} \quad n_1 + n_2 = n \\ C \in \mathbb{R}^{n \times n_2} \end{array}$$

all cols of M, B, C are lin ind.

rotation ish

cols C are $\perp$ to cols of B $\Rightarrow$ $C^T B = \begin{array}{c} n_1 \\ \left[\, 0 \,\right] \end{array} n_2$

$$M^{-1} = \frac{1}{\det(M)} \operatorname{Adj}(M)$$

$$\boxed{M^{-1} = \begin{bmatrix} (B^TB)^{-1}B^T \\ (C^TC)^{-1}C^T \end{bmatrix}}$$

$(B^TB)^{-1}B^T \quad [B] \qquad (C^TC)^{-1}C^T \quad [C]$

$M^{-1}M =$

$$\begin{bmatrix} (B^TB)^{-1}B^T \\ (C^TC)^{-1}C^T \end{bmatrix} \begin{bmatrix} B & C \end{bmatrix} = \begin{bmatrix} (B^TB)^{-1}B^TB & (B^TB)^{-1}\cancel{B^TC}^{\,0} \\ (C^TC)^{-1}\cancel{C^TB}^{\,0} & (C^TC)^{-1}C^TC \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

I

Stretch... $\quad M^{-1}M = I \Rightarrow MM^{-1} = I \Leftarrow$ fact

$$MM^{-1} = \begin{bmatrix} B & C \end{bmatrix} \begin{bmatrix} (B^TB)^{-1}B^T \\ (C^TC)^{-1}C^T \end{bmatrix} = \underbrace{B(B^TB)^{-1}B^T}_{\text{proj onto span of } B} + \underbrace{C(C^TC)^{-1}C^T}_{\text{proj onto span of } C} = I$$

not obvious

span of cols of B $\quad$ perpendicular $\quad$ span of cols of C

Define a coord transform related to the $R(A^T)$ & $N(A)$

constructed a basis for $N(A)$

$\Rightarrow$ cols of N

$$P = [A^T N]$$

assume A is full row rank and N is full col rank

$$M = [B\ C]$$

$$\underline{M^T M} = \begin{bmatrix} B^T \\ C^T \end{bmatrix} \begin{bmatrix} B & C \end{bmatrix} = \begin{bmatrix} B^T B & \cancel{B^T C}^{\,0} \\ \cancel{C^T B}^{\,0} & C^T C \end{bmatrix}$$

$$= \underline{\left[\begin{array}{c|c} B^T B & 0 \\ \hline 0 & C^T C \end{array}\right]}$$

$$\begin{bmatrix} B^T B & 0 \\ 0 & C^T C \end{bmatrix}^{-1} = \begin{bmatrix} (B^T B)^{-1} & 0 \\ 0 & (C^T C)^{-1} \end{bmatrix}$$

$$\underline{\begin{bmatrix} (B^T B)^{-1} & 0 \\ 0 & (C^T C)^{-1} \end{bmatrix}} M^T M = I$$

$$M^{-1} = \begin{bmatrix} (B^T B)^{-1} & 0 \\ 0 & (C^T C)^{-1} \end{bmatrix} \begin{bmatrix} B^T \\ C^T \end{bmatrix}$$

$$\underline{x} = P x' = \begin{bmatrix} A^T & N \end{bmatrix} \underline{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}} = A^T x_1' + N x_2'$$

$$x' = P^{-1} x$$

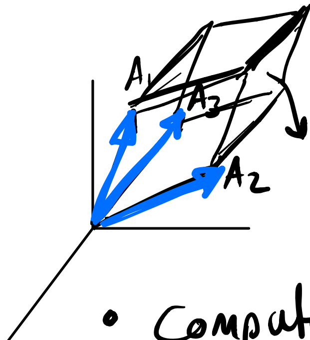
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} (AA^T)^{-1} A x \\ (N^T N)^{-1} N^T x \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} (AA^T)^{-1} A \\ (N^T N)^{-1} N^T \end{bmatrix}$$

EIGENVALUES & EIGENVECTORS:

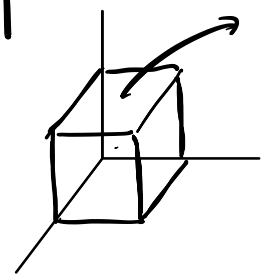
EIGEN: "SAME" in German

Review Determinants



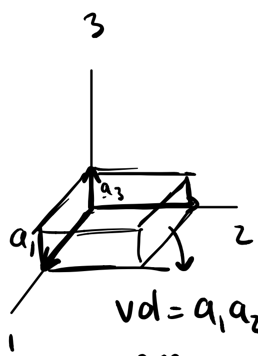
$A = [A_1 A_2 A_3]$

$\text{Vol} = |\det(A)|$



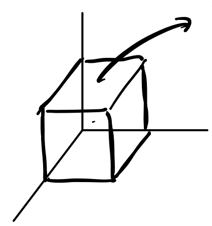
$\text{vol} = 1$

- Compute vol of parallelpipped $\text{Vol} = \left|\det\left([A_1 \, A_2 \, A_3]\right)\right|$

- 

$A = [A_1 A_2 A_3]$

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$



$\text{vol} = 1$

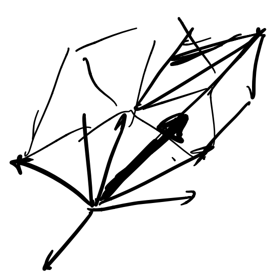
$\text{vol} = a_1 a_2 a_3$

$A \in \mathbb{R}^{n \times n}$ diagonal $\quad \det(A) = A_{11} \times \cdots \times A_{nn}$

- $\det(A) = 0$ A flattens out the unit cube

$\det(A) = 0$

A not full rank



EIGENVALUE PROBLEM: $A \in \mathbb{R}^{n \times n}$

find λ : scalar s.t. $\underline{A}\,\underline{v} = \underline{\lambda}\,\underline{v}$

v : vector