Review Question

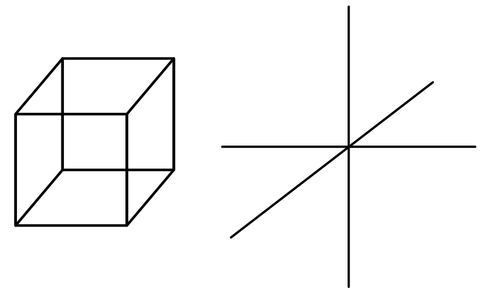
- BA

  B: invertible, A: fat

B invertible, A fat

$$\mathcal{N}(BA) = \mathcal{N}(A) \checkmark$$

$$x \in \mathcal{N}(A) \Rightarrow B\underbrace{Ax}_{0} = 0$$



$$\begin{bmatrix} -b_1^T- \\ \vdots \\ -b_n^T- \end{bmatrix} \underbrace{\begin{bmatrix} -a_1^T- \\ \vdots \\ -a_n^T- \end{bmatrix}}_{A} x$$

- AB

  A tall, B invertible

$$\mathcal{R}(AB) = \mathcal{R}(A)$$

$$\underbrace{\begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}}_{A} \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix}$$

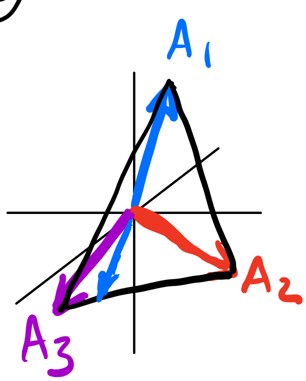
EIGENVALUE PROBLEM: $A \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$

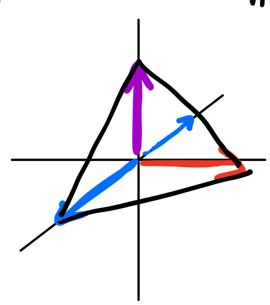
Eigen: "same" German

$\mathbb{R}^3$

$$A = [A_1 \, A_2 \, A_3]$$

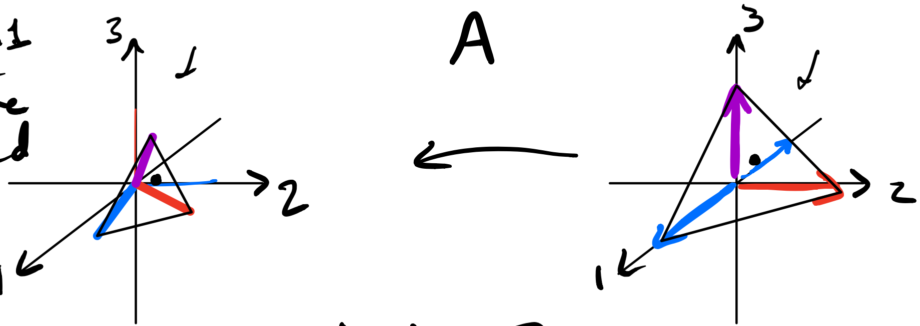
$\mathbb{R}^3$





Question 1

Is there a fixed line?



yes Perron Frobenius Thm

(contraction mapping thm
Brouwer's fixed pt. thm)

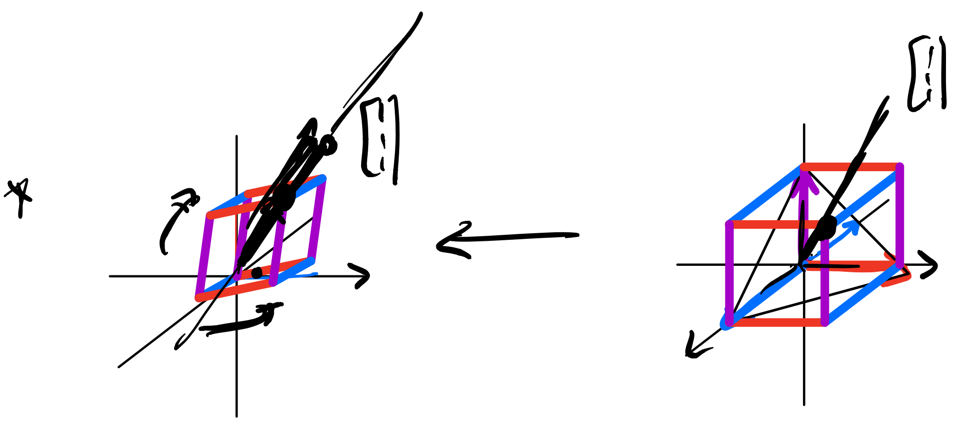
Question 2

How many fixed lines are there

$A \in \mathbb{R}^{3\times 3}$

at least 1 and up to 3 . .

n lines $\quad A \in \mathbb{R}^{n\times n}$



Eigenvalue Equation: $Av = \lambda v$ $\quad A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$ or $\boxed{\mathbb{C}}$

$v \in \mathbb{R}^n$

variables: v, λ

is this a linear system of eqns, could you solve w/ GE.

if λ given: $Av = \lambda v \Rightarrow \boxed{(A - \lambda I)v = 0}$

↳ find nullspace

$Av = \lambda \cdot v$ } → non linear in λ & v

$Av = \lambda v$

$Av - \lambda I v = 0$ $\Rightarrow$ $\boxed{(A - \lambda I)v = 0}$ ✱

is there a $v \neq 0$ that satisfies ✱ for every λ answer: no

for random λ: probably no $v \neq 0$

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first step: pick λ s.t. $(A - \lambda I)v = 0$ has a solution w/ $v \neq 0$

what is a condition we want for $A - \lambda I$?

$A - \lambda I$ not be full rank ←

↳ not invertible

↳ has a non-zero nullspace

↓

$\det(\underline{A - \lambda I}) = 0 \qquad A \in \mathbb{R}^{n \times n}$

Characteristic polynomial

n degree

$$\chi_A(s) = \det(A - sI) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$$

polynomial in s.
α's depend on the elements of A.

if λ is a root of $\chi_A(s)$ ie. $\chi_A(\lambda) = 0$

then $(A - \lambda I)$ has a non zero nullspace

$(A - \lambda I)v = 0$

$\Rightarrow$ interesting v.

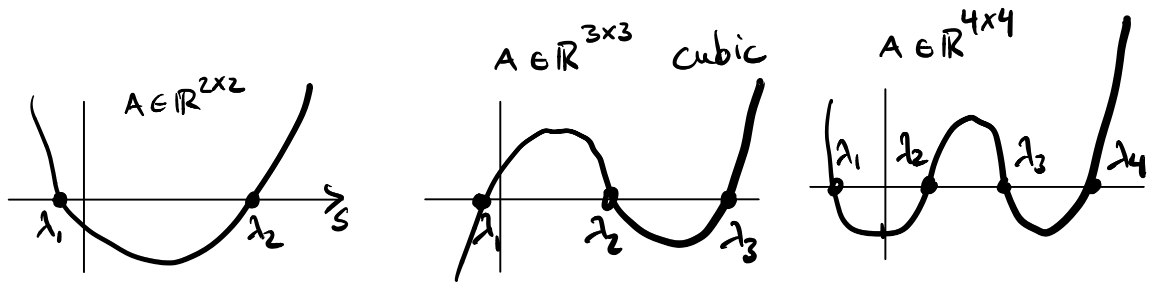
how many roots does $\chi_A(s)$ have?

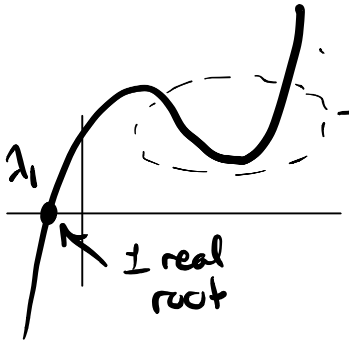
$\Rightarrow$ n roots (fundamental thm of algebra)

roots of $\chi_A(s)$: $\rho(A) = \{\underbrace{\lambda_1 \ldots \lambda_n}_{\text{eigen values}}\} =$ spectrum of A

Pictures

$\det(A - sI) = \chi_A(s)$





λ_1 : real

$e^{i\theta} + e^{-i\theta}$

$\lambda_2 = a + bi = re^{i\theta}$

$\lambda_3 = a - bi = re^{-i\theta}$

Complex roots of real polynomials come in conjugate pairs

rotations $e^{i\theta} = \cos\theta + \sin\theta\, i$

a real $A \in \mathbb{R}^{n \times n}$ can have complex eigenvalues → rotations

↳ always come in conjugate pairs

$\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) =$

$\frac{1}{2}\left(c\theta + \cancel{s\theta i} + c\theta - \cancel{s\theta i}\right)$

$= \cos\theta$

Rotation

- no fixed line...

→ plane of rotation

↑

span $\{V_2, V_3\}$

$\lambda_2 = a + bi$

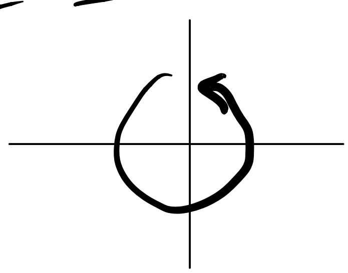
$\lambda_3 = a - bi$

→ $\lambda = c\theta + s\theta i$

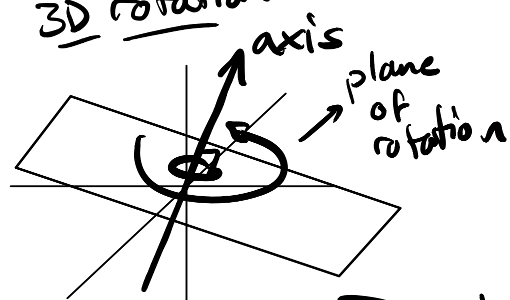
→ $\lambda^* = c\theta - s\theta i$

Side note:

2D rotation:



3D rotation



4D rotation:

either rotate about 1 plane

or

2 planes (perpendicular)

nD rotation:

n eigenvalues

↳ come in conjugate pairs

associated to a rotation in a plane

- n is even

  $\frac{n}{2}$ planes of rotation

- n is odd

  $\frac{n-1}{2}$ planes of rotation

Summary:

even dim rotations
$\rightarrow$ don't need to have an axis of rotation

odd dim rotations
$\rightarrow$ have to have an axis of rotation

n-1 eigenvalues
$\Rightarrow$ 1 left over
this eigenvalue has to be it's own conjugate
has to be real
for rotation matrices
has to be $\underline{0}$
axis of rotation

Eigenvectors

for eigenvalue λ_i: v_i is a right eigenvector if $Av_i = \lambda_i v_i$

note: eigenvector not super precise $\rightarrow$ "eigen subspace"
any vector in the subspace is an eigenvector

$\|A\underline{v_i} = \lambda_i \underline{v_i}\|$

if $A - \lambda_i I \rightarrow$ not full rank

$(A - \underline{\lambda_i I})\underline{v_i} = \underline{0}$

$\underline{w_i}^T (A - \lambda_i I) = 0$

for eigenvalue λ_i: w_i^T is a left eigenvector if $w_i^T A = \lambda_i w_i^T$

Matrix Diagonalization: $A \in \mathbb{R}^{n \times n}$ A diagonalizable

most matrices are diagonalizable

if A has n distinct eigenvalues

$\lambda_1 \neq \lambda_2 \neq \lambda_3$, etc...

then A is diagonalizable

if pick a random A it will be diagonalizable w/ probability 1

$$A v_i = \lambda_i v_i$$

$$[\underbrace{Av_1 \cdots Av_n}] = [\underbrace{\lambda_1 v_1 \cdots \lambda_n v_n}] \qquad D = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}$$

$$A [\underbrace{v_1 \cdots v_n}_{P}] = [\underbrace{v_1 \cdots v_n}_{P}] \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$AP = PD$ → eigenvalue equation for all eigenvalues

if A diagonalizable P is invertible

$$\boxed{A = PDP^{-1}}$$

⟹ A is similar to a diagonal matrix D w/ the eigenvalues on the diagonal

$$\underline{D} = P^{-1}AP$$

↳ diagonalizing A

what if we use left eigenvectors instead of right evecs.

$$\begin{bmatrix} w_1^T A \\ \vdots \\ w_n^T A \end{bmatrix} = \begin{bmatrix} \lambda_1 w_1^T \\ \vdots \\ \lambda_n w_n^T \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}}_{Q} A = D \begin{bmatrix} -w_1^T- \\ \vdots \\ -w_n^T- \end{bmatrix}$$

$$QA = DQ$$

$$\boxed{A = Q^{-1} D Q}$$

what is the relationship between Q & P

$$D = QAQ^{-1}$$

$$\underline{Q} = \underline{P^{-1}}$$

sort of... true if you pick v_i's and w_i's s.t.

$$w_i^T v_i = 1$$

$$A = PDP^{-1} = \begin{bmatrix} \underline{v_1 \cdots v_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^{-1}$$

$$P' = \begin{bmatrix} \underline{\alpha_1 v_1 \cdots \alpha_n v_n} \end{bmatrix} = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}$$

$$(P')^{-1} = \begin{bmatrix} 1/\alpha_1 & & \\ & \ddots & \\ & & 1/\alpha_n \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^{-1}$$

A,B diag

then $\underline{AB} = \underline{BA}$

$$A = P' D (P')^{-1} = \begin{bmatrix} \underline{v_1 \cdots v_n} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} 1/\alpha_1 & & 0 \\ & \ddots & \\ 0 & & 1/\alpha_n \end{bmatrix}}_{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} \begin{bmatrix} \underline{v_1 \cdots v_n} \end{bmatrix}^{-1}$$

$$= PDP^{-1}$$

A : diagonalizable

compute evals... $\lambda_1 \cdots \lambda_n$

compute right evecs $v_1 \cdots v_n$

$$A = PDP^{-1}$$

$$= \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{\text{cols are right evecs}} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^{-1}}_{\text{rows are left evecs}}$$