EIGENVALUES, EIGENVECTORS, DIAGONALIZATION

if A is diagonalizable ...

$$A = \underline{P}\, D\, \underline{P}^{-1}$$

A is related to D by a similarity transform.

$$= \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}}_{\text{right eigenvectors}} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\text{diagonal eigenvalues}} \underbrace{\begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}}_{\text{left eigenvectors}}$$

P^{-1}

$w_i^T v_i = 1$

$P^{-1}P = I$

$\rho(A)$: spectrum of A $\qquad \rho(A) := \{\lambda_1 \cdots \lambda_n\}$

Right eigenvector: $\quad A v_i = \lambda_i v_i$

Left eigenvector: $\quad w_i^T A = w_i^T \lambda_i$

Geometrically how do v_i's & w_i's relate

select w_i & v_i so that $w_i^T v_i = 1$

$$P^{-1}P = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} = \begin{bmatrix} w_1^T v_1 & \cdots & w_1^T v_n \\ w_n^T v_1 & \cdots & w_n^T v_n \end{bmatrix}$$

$$w_i^T v_j = 0 \quad j \neq i \quad \Leftarrow \quad = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & & 0 & 1 \end{bmatrix}$$

off diagonal terms are 0

a right eigenvector $\underline{v_i}$ is orthogonal to the subspace spanned by all other left eigenvectors $\underline{w_j}$ $j \neq i$

a left eigenvector $\underline{w_i}$ is orthogonal to the subspace spanned by all other right eigenvectors v_j $j \neq i$

$$Av_i = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - w_1^T - \\ \vdots \\ - w_n^T - \end{bmatrix} v_i \leftarrow$$

$$\begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \overset{1}{\cancel{w_i^T v_i}} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} w_1^T v_i \\ \vdots \\ w_n^T v_i \end{bmatrix}$$

$$\begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_i \\ 0 \end{bmatrix}$$

$$Av_i = \lambda_i v_i = v_1 0 + \cdots v_i \lambda_i \cdots + v_n 0$$

$$w_i^T A = w_i^T \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

$$= \begin{bmatrix} w_i^T v_1 \cdots w_i^T \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

$$\begin{bmatrix} 0 & \underset{1}{\underset{\parallel}{w_i^T v_i}} & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} \begin{bmatrix} w_1' \\ \vdots \\ w_n^T \end{bmatrix}$$

$$\begin{bmatrix} 0 & \lambda_i & 0 \end{bmatrix} \begin{bmatrix} - w_1^T - \\ - w_n^T - \end{bmatrix} = \lambda_i w_i^T$$

IN GENERAL: for eigenvector pairs
not true that $w_i^T v_i \neq 1$
eigenvectors really should be called
eigensubspaces $\rightarrow$ we don't know $|w_i|$ or $|v_i|$
can select w_i & v_i so that $w_i^T v_i = 1$

$$A = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - w_1^T - \\ \vdots \\ - w_n^T - \end{bmatrix}$$

$$\underline{A} = \sum_i v_i w_i^T \lambda_i = \sum_i \begin{bmatrix} v_i \end{bmatrix} \lambda_i \begin{bmatrix} - w_i^T - \end{bmatrix}$$

dyadic expansion

$$A v_i = \sum_j \begin{bmatrix} v_j \end{bmatrix} \lambda_j \underbrace{\begin{bmatrix} - w_j^T - \end{bmatrix} v_i}_{0 \text{ if } j \neq i}$$

EIGENVECTOR COORDINATES:

coords w.r.t the eigenvector basis

$$y = \underline{A} x \qquad x = Px' = \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{\text{basis of eigen vectors}} \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

writing x as a linear comb of right eigenvectors

$y = Py'$

$Py' = APx'$

$$Py' = PDP^{-1}Px' = PDx' \Rightarrow y' = Dx'$$

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \vdots \\ \lambda_n x_n' \end{bmatrix}$$

Powers of Matrices:

$$A = PDP^{-1}$$

$$A^k = \overbrace{A \times \cdots \times A}^{k}$$

$$= PDP^{-1} \times PDP^{-1} \times \cdots \times PDP^{-1}$$

$$A^k = PD^kP^{-1} = P \underbrace{\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}}_{D^k} P^{-1}$$

later

Polynomials of A:

$$f(X) = X^m + \alpha_{m-1} X^{m-1} + \cdots + \alpha_0 I$$

$$f(A) = PD^mP^{-1} + \alpha_{m-1} PD^{m-1}P^{-1} + \cdots + \alpha_0 I$$

$$= PD^{m}P^{-1} + \alpha_{m-1} PD^{m-1}P^{-1} + \cdots + \alpha_0 PP^{-1}$$

$$= P\left[D^m + \alpha_{m-1}D^{m-1} + \cdots + \alpha_0 I\right]P^{-1}$$

$$= P \begin{bmatrix} \lambda_1^m + \alpha_{m-1}\lambda_1^{m-1} + \cdots + \alpha_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m + \alpha_{m-1}\lambda_n^{m-1} + \cdots + \alpha_0 \end{bmatrix} P^{-1}$$

PREVIEW

Jordan FORM $A = PJP^{-1}$

MOST GENERAL VERSION OF DIAGONALIZING

$$J = \begin{bmatrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_n \end{bmatrix}$$

$$J^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$$

Remember: $\chi_A(s) = \det(sI - A) = \underbrace{s^n + \alpha_{n-1}s^{n-1} + \dots}_{} + \alpha_0$

$\underline{\chi_A(\lambda_i)} = 0$ λ_i's were roots of $\chi_A(s)$

$$\underline{\chi_A(A)} = P\underline{\chi_A(D)}P^{-1} = P\begin{bmatrix} \underline{\chi_A(\lambda_1)}_{0} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\chi_A(\lambda_n)}_{0} \end{bmatrix} P^{-1}$$

$$= P[0]P^{-1} \overset{0}{=} 0$$

$\chi_A(s)$ char poly: $\underline{\chi_A(A) = 0}$ Cayley-Hamilton Theorem

(for general matrices true but slightly different proof)

A result of Cayley hamilton $A \in \mathbb{R}^{n \times n}$ is that any polynomial function of any degree of a matrix can be written as a polynomial of degree $\underline{\underline{n-1}}$.

$\chi_A(s) = \det(sI - A) = \underbrace{s^n + \alpha_{n-1}s^{n-1} + \dots}_{} + \alpha_0$

$$|\underline{A^n}| = \underbrace{-\alpha_{n-1}A^{n-1} - \dots - \alpha_0 I}_{}$$

$$A^{n+2} = A^n A^2 = (\underline{-\alpha_{n-1}A^{n-1} - \cdots - \alpha_0 I})A^2$$

$$= -\underline{\alpha_{n-1}}A^{n+1} + \cdots$$

$$= -\alpha_{n-1}A^n A + \cdots$$

$$= -\alpha_{n-1}(-\underline{\alpha_{n-1}A^{n-1} - \cdots - \alpha_0 I})A$$

$$= -\alpha_{n-1}(-\alpha_{n-1}\underline{A}^n - \cdots)$$

$$= -\underline{\alpha_{n-1}}(-\underline{\alpha_{n-1}}(-\underline{\alpha_{n-1}A^{n-1} - \cdots - \alpha_0 I})$$

$$\chi_A(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \cancel{\alpha_0}^{\,0}$$

$$s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s = 0$$

$$(s^{n-1} + \alpha_{n-1}s^{n-2} + \cdots + \alpha_1)s = 0$$

$$\downarrow$$

$$\lambda_i = 0$$

Clever way to write A^{-1} as a polynomial? using $\chi_A(A) = 0$

$$\underline{\chi_A(A)} = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I = 0 \,]\, A^{-1}$$

$$A^{n-1} + \alpha_{n-1}A^{n-2} + \cdots + \alpha_1 I + \alpha_0 A^{-1} = 0$$

$$\boxed{A^{-1} = \frac{1}{\alpha_0}\left[-A^{n-1} - \alpha_{n-1}A^{n-2} + \cdots - \alpha_1 I\right]}$$

$\underline{A^{-1}}$: $A = PDP^{-1} \Rightarrow$

$$A^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1} = P\begin{bmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{bmatrix}P^{-1}$$

inverting a matrix is the same as inverting the eigenvalues square A is invertible iff $\lambda_i \neq 0 \ \forall i$ $\leftrightarrow \det(A) \neq 0$

Determinant & Trace — Eigenvalues

$$\det(A) = \det(PDP^{-1}) = \det(P)\det(D)\det(P^{-1})$$

$$= \cancel{\det(P)}\det(D)\frac{1}{\cancel{\det(P)}} = \det(D)$$

for diagonal matrix $\det(D)$ = product of diagonal

$$\boxed{\det(A) = \prod_{i=1}^{n} \lambda_i}$$

$$tr(A) = tr(PDP^{-1}) = tr(P^{-1}PD) = tr(D)$$

$$\boxed{tr(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i}$$

Taylor Expansions of Matrices...

Matrix Exponential:

$$e^{At} := I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

scalar: $\dot{x}(t) = \frac{dx}{dt} = \lambda x(t) \quad x(0) = x_0$

$x(t) \in \mathbb{R}$

$$x(t) = \text{const} \times e^{\lambda t} \qquad \frac{d}{dt}e^t = e^t$$

$$\dot{x} = \text{Const}\,\frac{d}{dt}e^{\lambda t} = \lambda \underbrace{\text{const}\,e^{\lambda t}}_{x(t)}$$

$$\text{Const} = x_0$$

Solution: $x(t) = e^{\lambda t}x_0$

vector case $x(t) \in \mathbb{R}^n$

$$\dot{x}(t) = \frac{dx}{dt} = Ax(t) \qquad x(0) = x_0$$

$$x(t) = e^{At}x_0$$

$$x(t) = Pe^{Dt}P^{-1}x_0$$

Change coords: $x(t) = Pz(t)$

writing x in the eigenvector coordinates

$$z(t) = e^{Dt}z(0)$$

$$z_1(t) = e^{\lambda_1 t}z_1(0)$$

$$\vdots$$

$$z_n(t) = e^{\lambda_n t}z_n(0)$$

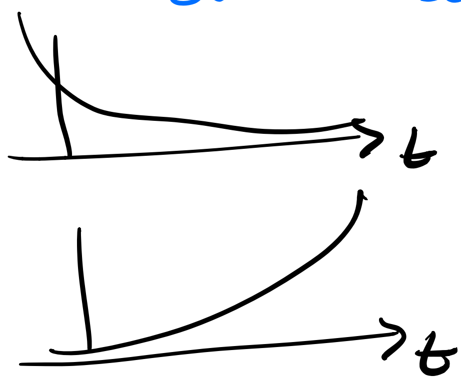
if $Re(\lambda_i) < 0$

$e^{\lambda_i t} \rightarrow 0$

if $Re(\lambda_i) > 0$

$e^{\lambda_i t} \rightarrow \infty$

in ODE's ea eigenvector evolves separately



Stability of linear systems

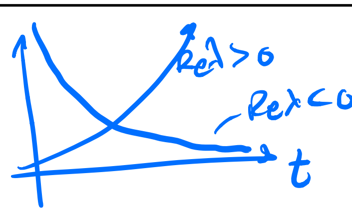
SUMMARY

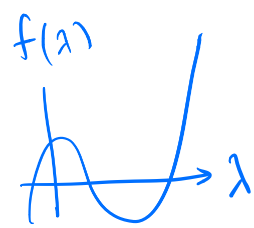
SPECTRAL MAPPING THM: $\sigma(A) = \{\lambda_1 \dots \lambda_n\}$

$f(\cdot)$: analytic function (polynomials, etc.)

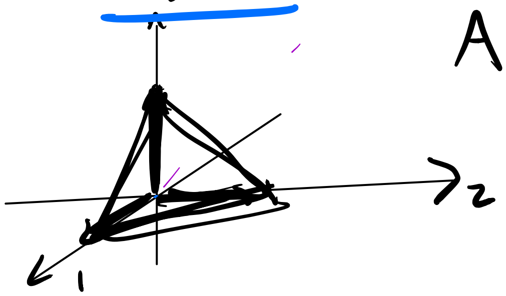
$$\sigma(f(A)) = \{f(\lambda_1) \dots f(\lambda_n)\}$$

$e^{\lambda t}$

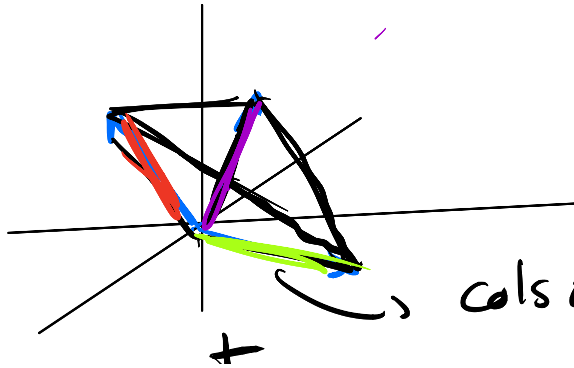




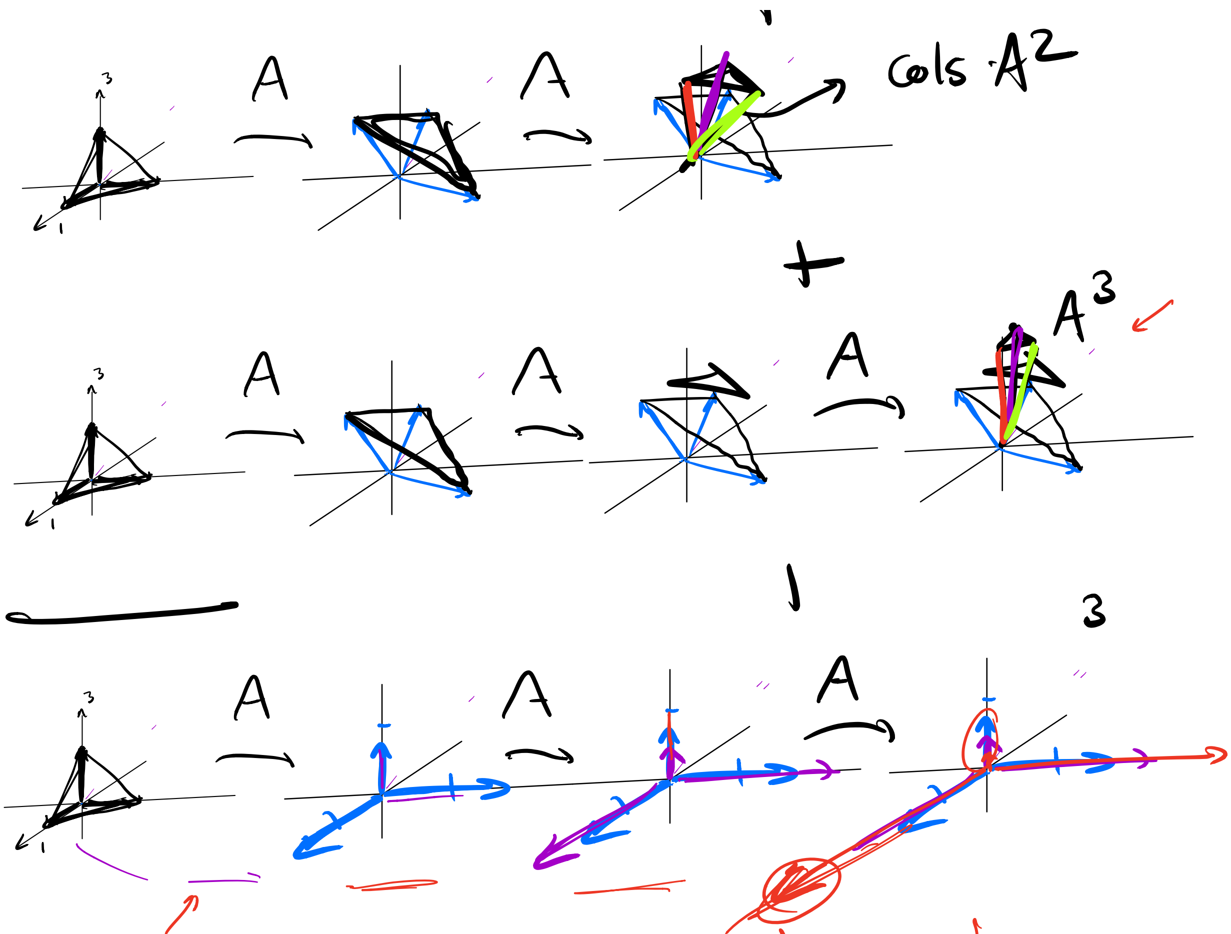
SCRIBBLES



A



cols of A



$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3$$