Eigenvalues & Vectors

- Complex evals & evecs] → related to rotations
- Nilpotent matrices $A^k = 0$
- Jordan Form — generalization of diagonalization

Review

Diagonalizable $\underline{A} \in \mathbb{R}^{n \times n}$ $\quad x \in \mathbb{R}^n$

$y = Ax$

$y' = A'\underline{x}'$

$$A = PDP^{-1}$$

$$= \begin{array}{ccc} P & D & P^{-1} \\ \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{\text{right evecs}} & \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} & \underbrace{\begin{bmatrix} - w_1^T - \\ \vdots \\ - w_n^T - \end{bmatrix}}_{\text{left evecs}} \end{array}$$

Mnemonic: "$A = RDL$"

$R = L^{-1}$

Note $A = PBP^{-1}$ → $B = \begin{bmatrix} * & & 0 \\ & * & \\ * & & \ddots \end{bmatrix}$ → still evals

↓

$A = \underline{PQ}DQ^{-1}P^{-1}$

P is not necessarily unique...

Ex. $P' = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} 1/|v_1| & & 0 \\ & \ddots & \\ 0 & & 1/|v_n| \end{bmatrix}}$

$$(P')^{-1} = \begin{bmatrix} |v_1| & & 0 \\ & \ddots & \\ 0 & & |v_n| \end{bmatrix} \begin{bmatrix} — w_1^T — \\ \vdots \\ — w_n^T — \end{bmatrix}$$

$$A = P' D (P')^{-1}$$

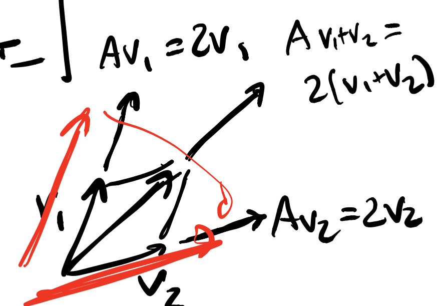
$$= \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \underline{\begin{bmatrix} 1/|v_1| & & 0 \\ & \ddots & \\ 0 & & 1/|v_n| \end{bmatrix}} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \underline{\begin{bmatrix} |v_1| & & 0 \\ & \ddots & \\ 0 & & |v_n| \end{bmatrix}} \begin{bmatrix} — w_1^T — \\ \vdots \\ — w_n^T — \end{bmatrix}$$

$$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} — w_1^T — \\ \vdots \\ — w_n^T — \end{bmatrix}$$

$$A = PDP^{-1} = P' D (P')^{-1}$$

Example : repeated evals ?

$$A = \underbrace{\begin{bmatrix} | & & & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & & & & | \end{bmatrix}}_{\text{2D right eigen subspace}} \underbrace{\begin{bmatrix} \begin{matrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{matrix} & & 0 \\ & \lambda_3 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}_{\downarrow} \begin{bmatrix} — w_1^T — \\ — w_2^T — \\ \vdots \\ — w_n^T — \end{bmatrix} \Big\} \rightarrow \text{2D left eigen subspace}$$



$$\begin{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} & v_3 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda I_{2\times2} & & 0 \\ & \lambda_3 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} \begin{bmatrix} — w_1^T — \\ — w_2^T — \end{bmatrix} \\ \vdots \\ — w_n^T — \end{bmatrix}$$

everything on plane is scaled by 2...

$$\begin{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} & v_3 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda Q Q^{-1} & & 0 \\ & \lambda_3 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} \begin{bmatrix} — w_1^T — \\ — w_2^T — \end{bmatrix} \\ \vdots \\ — w_n^T — \end{bmatrix}$$

$Q \in \mathbb{R}^{2\times2}$

$$\left[\underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix} Q}_{\text{different basis for that eigen subspace}} \; v_3 \cdots v_n \right] \begin{bmatrix} \lambda_1 I & & 0 \\ & \lambda_3 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} Q^{-1} \begin{bmatrix} - w_1^T - \\ - w_2^T - \end{bmatrix} \\ - w_3^T - \\ \vdots \\ - w_n^T - \end{bmatrix}$$

New set of eigen vectors...

right evecs ..

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$$

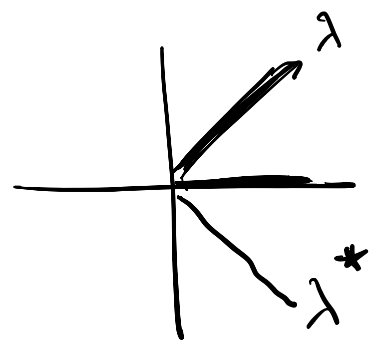
left evecs

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} - w_1^T - \\ \vdots \\ - w_n^T - \end{bmatrix}$$

Complex eigenvalues for $A \in \mathbb{R}^{n \times n}$

complex evals come in conjugate pairs...

$$\lambda = a + bi, \quad \lambda^* = a - bi$$
$$= re^{i\theta} \qquad = re^{-i\theta}$$



$$\sqrt{\lambda^* \lambda} = \sqrt{a^2 + b^2}$$

a useful matrix...

"carte. form" "polar form"

2x2 counter clockwise rotation

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix} = r \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

more precise

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = U \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} U^*$$

what is U...

$U \in \mathbb{C}^{2\times 2}$ unitary $U^{-1} = U^*$ $(U^*U = I)$

$$U = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$U^* = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$U^*U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} 1-i^2 & 1+i^2 \\ 1+i^2 & 1-i^2 \end{bmatrix} \frac{1}{2}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{2} = I$$

$$\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$\frac{1}{2} \begin{bmatrix} a+bi & a-bi \\ -ia+b & ia+b \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} 2a & \cancel{ia}-b-\cancel{ia}-b \\ -ia+b+ia+b & a+bi+a-bi \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} 2a & -2b \\ 2b & 2a \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Complex eigenvectors: $A \in \mathbb{R}^{n\times n}$ also come in conjugate pairs

$$A = \begin{bmatrix} v_1 & v_2 & \cdots \end{bmatrix} \begin{bmatrix} \lambda & 0 & & \\ 0 & \lambda^* & & \\ & & \ddots & \end{bmatrix} \begin{bmatrix} - & w_1^* & - \\ - & w_2^* & - \\ & \vdots & \end{bmatrix}$$

use $\bar{v}_1$ to mean conjugate ea. term of v_1
(without transposing)

$\underline{v_2 = \bar{v}_1}$ similarly $w_2 = \bar{w}_1$

$P \in \mathbb{C}^{n \times n}$

$$\overset{P}{\begin{bmatrix} v_1 \; v_2 & \cdots \end{bmatrix}} \begin{bmatrix} a+bi & 0 & \\ 0 & a-bi & \\ & & \ddots \end{bmatrix} \overset{P^{-1}}{\begin{bmatrix} - w_1^* - \\ - w_2^* - \end{bmatrix}}$$

$$\begin{bmatrix} v_1 v_2 & \cdots \end{bmatrix} \begin{bmatrix} u^* \begin{bmatrix} a & -b \\ b & a \end{bmatrix} u & 0 \\ 0 & \ddots \end{bmatrix} \begin{bmatrix} - w_1^* - \\ - w_2^* - \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} v_1 v_2 \,|\, u^* & v_3 \cdots v_n \end{bmatrix}}_{P \begin{bmatrix} u^* & \\ & I \end{bmatrix}} \begin{bmatrix} a & -b & & 0 \\ b & a & & \\ & 0 & & \ddots \end{bmatrix} \underbrace{\begin{bmatrix} u \begin{bmatrix} - w_1^* - \\ - w_2^* - \end{bmatrix} \\ \\ \end{bmatrix}}_{\begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} P^{-1}}$$

$$\underbrace{\begin{bmatrix} v_1 \; v_2 \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \frac{1}{\sqrt{2}}}_{\begin{bmatrix} \frac{1}{\sqrt{2}}(v_1+v_2) & \frac{i}{\sqrt{2}}(v_1-v_2) \end{bmatrix}} \qquad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} w_1^* \\ w_2^* \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}}(w_1^* + w_2^*) \\ \frac{i}{\sqrt{2}}(w_2^* - w_1^*) \end{bmatrix}$$

real

since $v_2 = \bar{v}_1$ and $w_2 = \bar{w}_1$

real

$$p_1 = \frac{1}{\sqrt{2}}(v_1 + v_2) \qquad q_1 = \frac{1}{\sqrt{2}}(w_1 + w_2)$$

$$p_2 = \frac{i}{\sqrt{2}}(v_1 - v_2) \qquad q_2 = \frac{i}{\sqrt{2}}(w_2 - w_1)$$

$p_1, p_2, q_1, q_2 \in \mathbb{R}^n$

$v_1, v_2, w_1, w_2 \in \mathbb{C}^n$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots \end{bmatrix} \begin{bmatrix} a+bi & 0 & \\ 0 & a-bi & \\ & & \ddots \end{bmatrix} \begin{bmatrix} - & w_1^* & - \\ - & w_2^* & - \\ & \vdots & \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} p_1 & p_2 & \cdots \end{bmatrix}}_{P'} \begin{bmatrix} \begin{matrix} a & -b \\ b & a \end{matrix} & 0 \\ 0 & \ddots \end{bmatrix} \underbrace{\begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \end{bmatrix}}_{(P')^{-1}} \qquad P' \in \mathbb{R}^{n \times n}$$

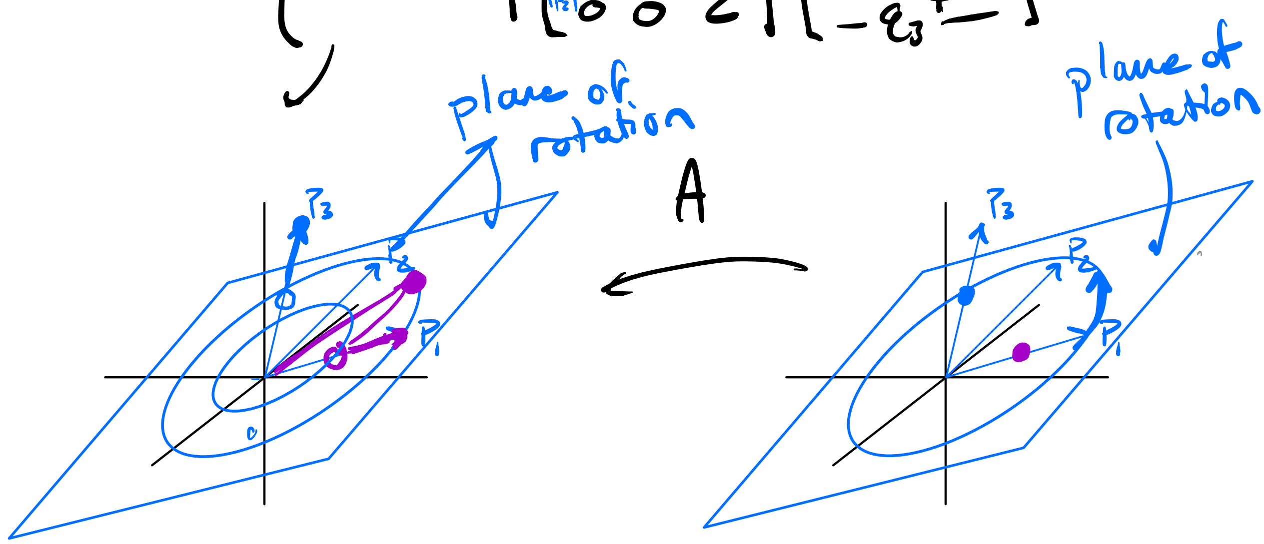
$$= \begin{bmatrix} p_1 & p_2 & \cdots \end{bmatrix} \begin{bmatrix} r \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} & 0 \\ 0 & \ddots \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \end{bmatrix}$$

Example $A \in \mathbb{R}^{3\times 3}$ $\qquad \theta = \pi/4$

$$\begin{array}{c} \frac{1}{|p_1|} \quad \frac{1}{|p_2|} \quad \frac{|p_2|}{|p_1|} \end{array}$$

$$A = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} 2 \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 2 \end{bmatrix} \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ - & q_3^T & - \end{bmatrix}$$

$\frac{|p_1|}{|p_2|}$



for a circular rotation ...

p_1 & p_2 are $\perp$

relative position of p_1 & p_2
gives shape of elliptical rotation...

for diff eq. $\dot{x} = Ax$ $\quad \underline{x(t)} = \underbrace{e^{At}}_{\substack{\text{same} \\ \text{eigen} \\ \text{structure} \\ \text{as } A}} x(0)$

spectral mapping thm:

$$\underline{A^k} = \underline{P D^k P^{-1}}$$

let $A = P r \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} P^{-1}$

now $A^k = P r^k \begin{bmatrix} ck\theta & -sk\theta \\ sk\theta & ck\theta \end{bmatrix} P^{-1}$

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} = \begin{bmatrix} c\theta^2 - s\theta^2 & -2c\theta s\theta \\ 2c\theta s\theta & c\theta^2 - s\theta^2 \end{bmatrix}$$

$$= \begin{bmatrix} c2\theta & -s2\theta \\ s2\theta & c2\theta \end{bmatrix}$$

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} e^{i2\theta} & 0 \\ 0 & e^{-i2\theta} \end{bmatrix}$$

what A gives a rotation

$\dot{x} = Ax$

$x(t) = e^{At} x(0)$

want eigenvalues of $e^{\lambda t}$ have form $e^{i\theta}, e^{-i\theta}$

$e^{i\theta} = e^{\lambda t} \quad e^{-i\theta} = e^{\lambda^* t}$

means that evals of A

$\lambda = \frac{\theta}{t} i \quad \lambda^* = \frac{-\theta}{t} i$

$\dot{x} = Ax$ is a rotation if A has purely imaginary eigenvalues

A is skew symmetric $A = -A^T$ $\Leftarrow$