JORDAN FORM – GENERALIZATION OF DIAGONALIZATION

any matrix

diagonalizable matrix :
$$A = PDP^{-1}$$
$$\begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^{-1}$$

general matrix:
$$A = PJP^{-1}$$
$$J \downarrow \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & \ddots \end{bmatrix} \qquad J_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ & \ddots & \\ 0 & & \lambda_1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \lambda_2 & 1 & 0 \\ & \ddots & \\ 0 & & \lambda_2 \end{bmatrix}$$

this shows up because of

- repeated eigenvalues
  eigenvectors → live in subspaces that are in 2D or higher
  → ambiguity in what eigenvectors are
- nilpotent matrices

Nilpotent Matrix: $A \in \mathbb{R}^{n \times n}$

$$A^k = 0 \longrightarrow \text{no analog w/ numbers}$$

EX.
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$$

$$\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex not nilpotent $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

Ex $\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$

Ex $\left(P\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}P^{-1}\right)^2 = P\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\underline{P^{-1}P}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}P^{-1}$

$$= P\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}P^{-1}$$

$$\left(P\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}P^{-1}\right)^3 = P\,0\,P^{-1} = 0$$

What are the eigenvectors of a nilpotent matrix?

Ex. $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} \rightarrow \lambda = 0$ ✗

$\rightarrow \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}*\\*\\*\end{bmatrix} = \begin{bmatrix}*\\*\\0\end{bmatrix}$ only possible eigenvalue is 0

$\hookrightarrow P\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}P^{-1} = [0]$

$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_2\\x_3\\0\end{bmatrix}$ $\lambda = 0$ $x_2 = 0\cdot x_1$

$x_3 = 0\cdot x_2$

$x_2 = 0$ $x_3 = 0$

$\lambda = 0$ $v = \begin{bmatrix}1\\0\\0\end{bmatrix}$ ← only true evec

only eval

$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Characteristic

$\det\begin{bmatrix}s&-1&0\\0&s&-1\\0&0&s\end{bmatrix} = s^3 \longrightarrow$ repeated eigenvalue of 0

$$\begin{vmatrix} s-\lambda & -1 & \\ & s-\lambda_2 & -1 \\ & & s-\lambda_3 \end{vmatrix} \Rightarrow (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)$$

A repeated eigenvalue of 0, but only 1 <u>eigenvector</u>

Wanted to construct a basis for the space from eigenvectors but we didn't have enough of them - needed 3 only have 1

what are eigenvalues of

$$\nearrow\ A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \det \begin{bmatrix} s-\lambda & -1 & 0 \\ 0 & s-\lambda & -1 \\ 0 & 0 & s-\lambda \end{bmatrix} = \underline{\underline{(s-\lambda)^3}}$$

repeated eval of λ...

what is the eigenvector associated w/ $\underline{\lambda}$ → $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ✓

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\uparrow}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \in N(\underline{\underline{\lambda I - A}}) = N(\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}})$$

only has a 1D nullspace

$$N((\lambda I - A)^2)$$

$$N\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{ \underset{\substack{\uparrow \\ \text{orig evec}}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$N((\lambda I - A)^3) = N(0) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

if an eigenvalue λ is repeated k times...

and $\dim N((\lambda I - A)) < k$

then we can look at vectors in

$$N((\lambda I - A)^l) \text{ for } l \leq k$$

vectors in this nullspace are called <u>generalized</u> <u>eigenvectors</u>

for $l = 1$: first order generalized evecs
or just eigenvectors

for $l = 2$: 2nd order generalized evecs...

for $l = 3$: 3rd order etc...

if no repeated eigenvalues...

$\chi_A(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

$\mathcal{N}(\underline{\lambda_i I - A})$ → has to be at least 1 dimension

$\det(\underline{\lambda_i I - A}) = 0$

Jordan Form:

$$A = \begin{bmatrix} v_1 & \cdots & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix}$$

for each Jordan block we can find at least 1 eigenvector v_i

then find generalized eigenvectors for ea. repeated eigenvalue $\mathcal{N}((\lambda_i I - A)^l)$

w_1, u_1, etc...

$$A = \overset{P}{\begin{bmatrix} v_1 & w_1 & u_1 & \cdots \end{bmatrix}} \left[\begin{array}{c|c} \begin{matrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{matrix} & 0 \\ \hline 0 & J_k \end{array}\right] \begin{bmatrix} P^{-1} \end{bmatrix}$$

v_1: regular eigen vector

w_1: 2nd order eigen vector

u_1: 3rd order eigen vector

$$\underline{(\lambda_1 I - A)} v_1 \checkmark = \begin{bmatrix} v_1 & w_1 & u_1 & \cdots \end{bmatrix} \underbrace{\begin{bmatrix} \boxed{0} & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ & & & * & \\ & & & & * \end{bmatrix}}_{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}} \underbrace{\begin{bmatrix} P^{-1} \end{bmatrix} v_1}_{\to \begin{bmatrix} \boxed{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} =$$

$v_1 \in N(\lambda_1 I - A)$

$$\underline{(\lambda_1 I - A)^2} = \begin{bmatrix} v_1 & w_1 & u_1 & \cdots \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & * & \\ & & & & * \end{bmatrix} \begin{bmatrix} P^{-1} \end{bmatrix} w_1$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$v_1, w_1 \in$
$N((\lambda_1 I - A)^2)$

$$\underline{(\lambda_1 I - A)^3} = \begin{bmatrix} v_1 & w_1 & u_1 & \cdots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & * & \\ & & & & * \end{bmatrix} \begin{bmatrix} P^{-1} \end{bmatrix} u_1$$

v_1, w_1, u_1
$N((\lambda_1 I - A)^3)$

$$A = \underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{\text{generalized right evecs}} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{\text{generalized left evecs}}$$

$J_i = \lambda_i I + N_i$

k is # of generalized evecs associated with λ_i

$$N_i = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & \ddots & 1 \\ 0 & & 0 \end{bmatrix}$$

$N_i \in \mathbb{R}^{k \times k}$

Characteristic polynomial ✓

$$\chi_A(s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$$= \underline{(s-\lambda_1)}^{k_1} \underline{(s-\lambda_2)}^{k_2} \cdots \underline{(s-\lambda_p)}^{k_p}$$

k_i is # of times λ_i is repeated as an eigenvalue

l_i is $\dim N(\underline{\lambda_i I - A})$

$$\dim N(\underline{\lambda_i I - A}) = l_i \leq k_i$$

ex.

$$\dim N((\lambda_i I - A)^2) = 5 \qquad \begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} & & \\ & [\lambda_1] & \\ & & \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \end{bmatrix} \qquad k_1 = 6$$

$$l_1 = 3$$

$$\dim N((\lambda_i I - A)^3) = 6 \qquad \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & & \\ & [0] & \\ & & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Minimal polynomial:

$$(s-\lambda_1)^{k_1-l_1} \cdots\cdots (s-\lambda_p)^{k_p-l_p}$$

l_i: # of regular eigenvectors for λ_i

$k_i - l_i$: # of generalized eigenvectors for λ_i

$$\begin{bmatrix} v_1 & w_1 & u_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} v_1 & w_1 & u_1 \end{bmatrix}^{-1} \begin{bmatrix} w_1 \end{bmatrix}$$

$$v_1 + \lambda_1 w_1 = \qquad \begin{bmatrix} 1 \\ \lambda_1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} v_1 & w_1 & u_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} v_1 & w_1 & u_1 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \end{bmatrix}$$

$$w_1 + \lambda_1 u_1 = \qquad \begin{bmatrix} 0 \\ 1 \\ \lambda_1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{Av_1 = \lambda_1 v_1} \qquad \boxed{Aw_1 = v_1 + \lambda_1 w_1} \qquad \boxed{Au_1 = \lambda_1 u_1 + w_1}$$

1st order gen evecs — 2nd order — 3rd order

$(A - \lambda_1 I) v_1 = 0$

$(A - \lambda_1 I) w_1 = v_1 \implies (A - \lambda_1 I)^2 w_1 = 0$

$(A - \lambda_1 I) u_1 = w_1 \implies (A - \lambda_1 I)^2 u_1 = v_1$

$(A - \lambda_1 I)^3 u_1 = 0$

generalized eigenvectors come in chains $u_1 \longrightarrow w_1 \longrightarrow v_1$

Relationships between left & right gen. evecs

diagonalizable case:

$$\overset{P}{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - q_1^T - \\ \vdots \\ - q_n^T - \end{bmatrix}$$

left & right evecs come in pairs

$$v_1, q_1^T \quad \cdots \quad v_n, q_n^T$$

Jordan form case:

$$\underset{\begin{array}{ccc} \uparrow & \uparrow & \nwarrow \\ \text{1st order right gen evec} & \text{2nd} & \text{3rd} \end{array}}{\begin{bmatrix} | & | & | \\ v_1 & w_1 & u_1 \\ | & | & | \end{bmatrix}} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} - q_1^T - \\ - p_1^T - \\ - s_1^T - \end{bmatrix} \overset{\swarrow}{\begin{array}{l} \leftarrow \text{3rd order} \\ \leftarrow \text{2nd order} \\ \leftarrow \text{1st order left evec} \end{array}}$$

$$s_1^T \begin{bmatrix} | & | & | \\ v_1 & w_1 & u_1 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & \overset{\swarrow}{0} \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} - q_1^T - \\ - p_1^T - \\ - s_1^T - \end{bmatrix}$$

$$[0\ 0\ 1] \rightarrow [0\ 0\ \lambda_1] = \lambda_1 s_1^T$$

$$p_1^T \begin{bmatrix} | & | & | \\ v_1 & w_1 & u_1 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} - q_1^T - \\ - p_1^T - \\ - s_1^T - \end{bmatrix}$$

$$[0\ 1\ 0] \rightarrow [0\ \lambda_1\ 1] = \lambda_1 p_1^T + s_1^T$$

switching order of evecs :

$$\begin{array}{cccc} & \downarrow & & \\ \begin{bmatrix} | & | & | \\ u_1 & w_1 & v_1 \\ | & | & | \end{bmatrix} & \begin{bmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_1 \end{bmatrix} & \begin{bmatrix} - s_1^T - \\ - p_1^T - \\ - q_1^T - \end{bmatrix} & \begin{array}{l} \swarrow \text{1st} \\ \swarrow \text{2nd} \\ \swarrow \text{3rd} \end{array} \\ \begin{array}{ccc} \swarrow & \uparrow & \uparrow \\ \text{3rd} & \text{2nd} & \text{1st} \end{array} & \nearrow & \uparrow & \end{array}$$

$\textcircled{?}$

$$\begin{array}{cccc} & \swarrow & & \\ \begin{bmatrix} | & | & | \\ u_1 & v_1 & w_1 \\ | & | & | \end{bmatrix} & \begin{bmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} & \begin{bmatrix} - s_1^T - \\ - q_1^T - \\ - p_1^T - \end{bmatrix} & \begin{array}{l} \swarrow \text{1st} \\ \swarrow \text{3rd} \\ \swarrow \text{2nd} \end{array} \\ \begin{array}{ccc} \swarrow & \uparrow & \uparrow \\ \text{3rd} & \text{1st} & \text{2nd} \end{array} & & \uparrow & \end{array}$$

length of gen evec

$$\begin{bmatrix} v_1 & w_1 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1/\alpha_1 & 0 \\ 0 & 1/\beta_1 \end{bmatrix}} \begin{bmatrix} - q_1^T - \\ - p_1^T - \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \alpha_1 \frac{1}{\alpha_1} & \alpha_1/\beta_1 \\ 0 & \lambda_1 \beta_1 1/\beta_1 \end{bmatrix} \qquad \begin{bmatrix} \lambda_1 & \alpha_1/\beta_1 \swarrow \\ 0 & \lambda_1 \end{bmatrix}$$

Spectral Mapping Thm : $\swarrow$ $f(A)$

$$A = PJP^{-1} \implies A^k = PJ^kP^{-1}$$

diagonal: $D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$

Jordan: $\begin{bmatrix} J_1^k & & \\ & \ddots & \\ & & J_p^k \end{bmatrix}$

$\swarrow$

for applying polynomial functions $f(J_i)$ $f(s)$

$$J_i = \begin{bmatrix} \lambda_i & \overset{\downarrow}{1} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \quad n_i \quad J_i \in \mathbb{R}^{n_i \times n_i}$$

$$f(J_i) = n_i \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{f''(\lambda_i)}{2} & \rightarrow & \frac{f^{(n_i-1)}(\lambda)}{(n_i-1)!} \\ & \ddots & \ddots & \ddots & \frac{f''(\lambda_i)}{2} \\ & & \ddots & \ddots & f'(\lambda_i) \\ 0 & & & & f(\lambda_i) \end{bmatrix}$$

$$f'(\lambda_i) = \frac{\partial f}{\partial s}\bigg|_{\lambda_i} \quad f''(\lambda_i) = \frac{\partial^2 f}{\partial s^2}\bigg|_{\lambda_i} \quad \text{etc...}$$

$$J_i^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{n_i-1}\lambda_i^{k-(n_i-1)} \\ & \ddots & \binom{k}{1}\lambda_i^{k-1} \\ 0 & & \lambda_i^k \end{bmatrix}$$

Summary:

if you have repeated eigenvalues...

- eigen subspaces are greater than 1D....
- nilpotent matrix structure added into an eigen subspace

$$\lambda_i I \longrightarrow \underbrace{\lambda_i I + N_i}_{\hookrightarrow \text{ creates generalized eigenvectors}}$$

- generalized evecs u_i have different orders depending on the power l_i s.t. $u_i \in \mathcal{N}\left((\lambda_i I - A)^{l_i}\right)$
- gen evecs come in chains...
- chains are reversed for left & right gen evecs.
- Jordan Form

$$A = PJP^{-1} \quad J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix}$$

Rest of Quarter:

- Linear vector fields & e^{At}
- $\boxed{\text{SVD}}$
- other decompositions
  - polar → sym, skew sym matrices
  - QR - gram schmidt
  - LU
- Positive definite matrices
  (congruent transformations)
  semi definite programming
- Schur complements
- sparsity → convex relaxation

Numerics for CVX
- gradient descent
- conjugate gradient descent
- simplex method
- interior pt. methods

- quaternions
  vs. rotations
  SLERP
  gimbal lock
- affine transformation

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homogeneous trans.

SO(3) : rotations
Lie group

so(3) : skew sym matrices
Lie algebra

$$se(3) = \left[\begin{array}{c|c} R & p \\ \hline * * * & 1 \end{array}\right]$$

rotations & trans.

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

dual quaternions

- Matrix group theory
  ⇒ rotations SO(n), SU(n)
  Lie group, Lie algebra } → robotics

- Numerical linear algebra
  evd, svd, Gram schmidt

Markov chains
MDP's
Graphs
Laplacians

Householder reflections