Eigenvalues & Stability:

$A \in \mathbb{R}^{n \times n}$ diagonalizable

$$A = \underbrace{P}_{\text{cols right evecs}} \underbrace{D}_{\text{evals}} \underbrace{P^{-1}}_{\text{rows left evecs}}$$

Spectral Mapping Thm:

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} P^{-1}$$

polynomials $f(A) = \alpha_1 A^{k_1} + \alpha_2 A^{k_2} \cdots$

$$= P(\alpha_1 D^{k_1} + \alpha_2 D^{k_2} + \cdots) P^{-1}$$

analytic functions: $(\cdot)^{-1}$ $\sqrt{(\cdot)}$

$$(A^{1/2})(A^{1/2}) = A \qquad A^{1/2} = P \begin{bmatrix} \lambda_1^{1/2} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{1/2} \end{bmatrix} P^{-1}$$

Matrix Exponential

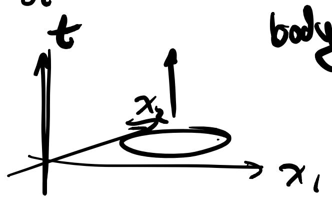
$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

useful for solving ORDINARY DIFF EQs (ODEs)

ODE:

$\frac{d}{dt}x$ $\dot{x}$

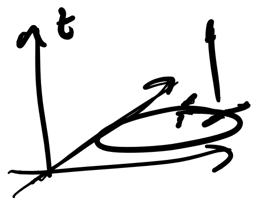
time evolution of rigid body sys.



PDE:

$\frac{d}{dt}$ $\frac{d}{dx_1}$ $\frac{d}{dx_2}$

time evolution of flexible systems



SIDE NOTE

ODES:

scalar sys: $\dot{x} = \lambda \Rightarrow x(t) = \lambda t$

$\underline{\dot{x} = \boxed{\lambda x}} \rightarrow$ RHS is linear in the state.

linear diff eqn

$\dot{x} = \underline{\lambda(t)}\, x$

$\dot{x} = \lambda x$

Linear time invariant sys (LTI)

$\dot{x} = \lambda(t) x$

linear time varying system (LTV)

$\underline{\dot{x} = \lambda x}$

why is e^t special?

$\frac{d}{dt} \underline{e}^t = e^t$

$\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

$x(t) = \text{const}\, e^{\lambda t}$

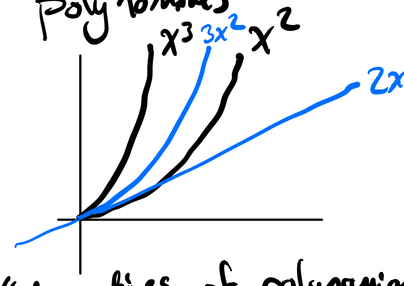
at time $t = 0$

$x(0) = \text{const}\ 1.$

$x(t) = e^{\lambda t} x(0)$

Derivatives:

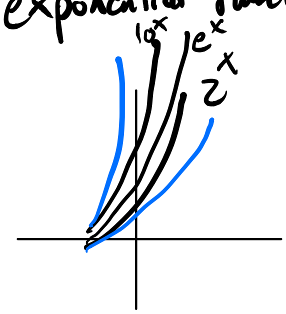
polynomials



"derivatives of polynomials don't grow as fast as the original polynomial"

"derivatives of poly have more shallow slopes

exponential functions



$e = 2.7\_\_\_$

Note: $\frac{d}{dt}$ linear operator on functions

$f(t)$: inf. dim vec $\quad \frac{d}{dt}$: lin operator on $f(t)$

$e^{\lambda t}$: eigenfunctions of $\frac{d}{dt}$ with eigenvalue λ

<u>fin dim analog</u> $\quad$ <u>inf dim case</u>

$$Mf = \lambda f \qquad \frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}$$

<u>Matrix Case:</u> $A \in \mathbb{R}^{n \times n}$ diagonalizable

$$\rightarrow \dot{x} = Ax \qquad A = PDP^{-1}$$

$$\dot{x} = PDP^{-1}x \qquad x = Pz \rightarrow$$ writing x in terms of a basis of right evecs

$$P\dot{z} = PDP^{-1}Pz$$

$$\dot{z} = Dz \implies \dot{z}_1 = \lambda_1 z_1 \implies z_1(t) = e^{\lambda_1 t} z_1(0)$$

$$\vdots$$

$$\dot{z}_n = \lambda_n z_n \implies z_n(t) = e^{\lambda_n t} z_n(0)$$

$$x(t) = e^{At}x(0) = Pe^{Dt}\underbrace{P^{-1}x(0)}_{z(0)}$$

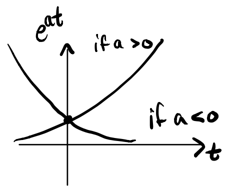
$$\underbrace{P^{-1}x(t)}_{z(t)} = e^{Dt} z(0)$$

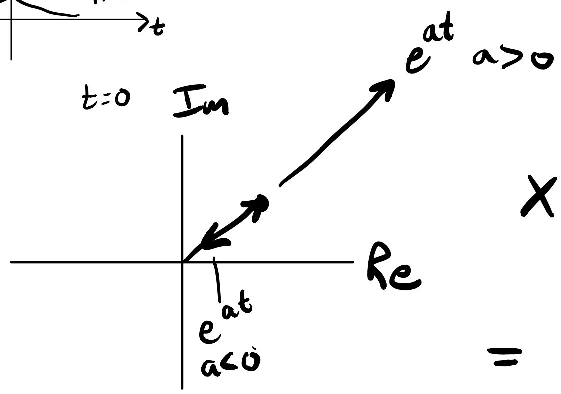
Eigenvalue of A, $\lambda = a+bi$

what are evals of e^{At}?

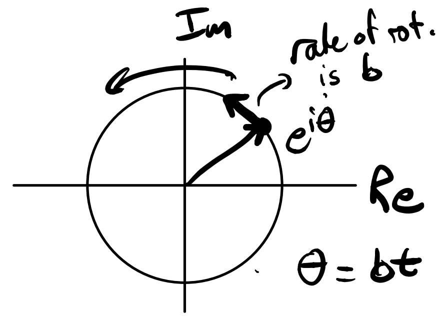
$\lambda \in \rho(A)$ $\qquad$ $\mu \in \rho(e^{At})$

$$\mu = e^{\lambda t} = e^{(a+bi)t} = \underline{e^{at}}\ \underline{e^{bit}}$$



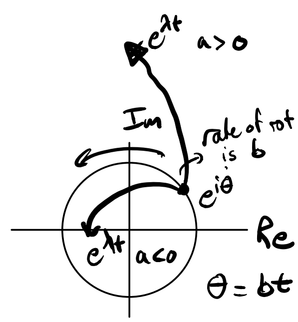


$\times$



$=$

$$e^{\lambda t} = \underbrace{e^{at}}_{\text{mag}}\ \underbrace{e^{bit}}_{\text{phase or orientation}}$$



<u>Stability</u> <u>cond</u>: $\dot{x} = Ax$

a linear sys is stable if $Re(\lambda) < 0$ for $\lambda \in \rho(A)$

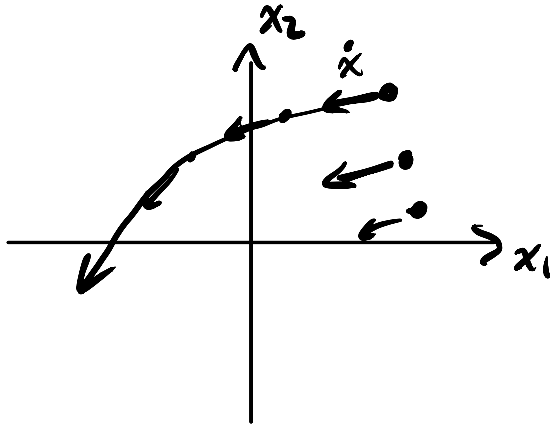
linear sys is marginally stable if $Re(\lambda) = 0$ for some λ

a linear sys is unstable if $Re(\lambda) > 0$ for $\lambda \in \rho(A)$

Quiver Plots

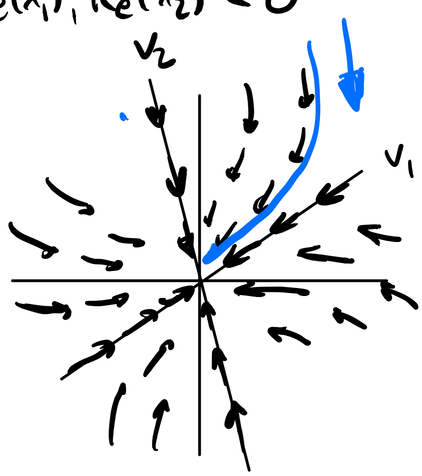
$\dot{x} = Ax$ } vector field

at ea point x. have a vector that tells us the direction of evolution

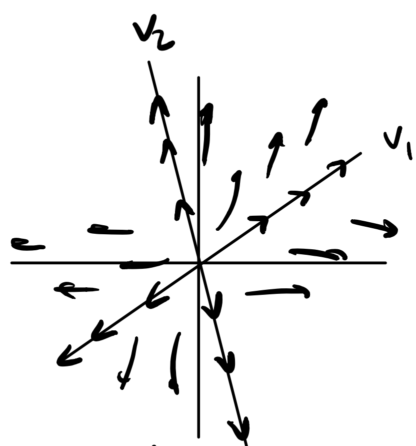


$A \in \mathbb{R}^{2\times 2}$ $\lambda_1, \lambda_2 \in \rho(A)$ evecs v_1, v_2

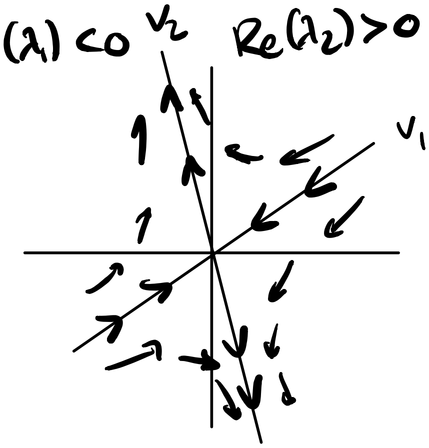
$Re(\lambda_1), Re(\lambda_2) < 0$



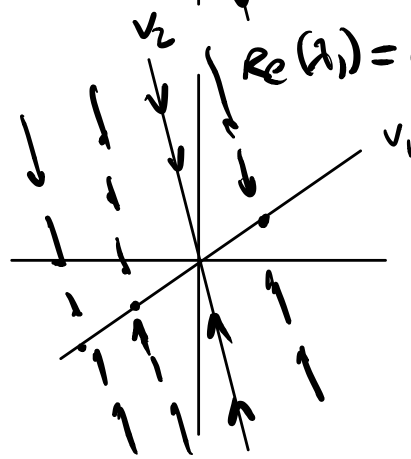
$Re(\lambda_1), Re(\lambda_2) > 0$



$Re(\lambda_1) < 0$ $Re(\lambda_2) > 0$

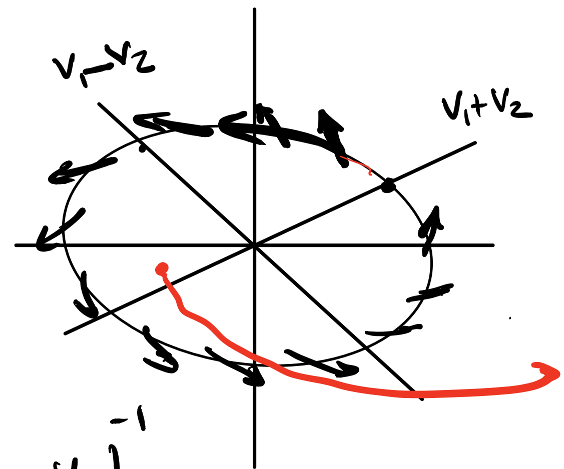


$Re(\lambda_1) = 0$ $Re(\lambda_2) < 0$



$A \in \mathbb{R}^{2\times 2}$ $\lambda, \lambda^* \in \rho(A)$ complex $a > 0$

$$A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$



can't draw v_1, v_2

interested in $x(t) = e^{At} x(0)$

$$e^{At} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{at} e^{bit} & 0 \\ 0 & e^{at} e^{-bit} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} e^{at} \begin{bmatrix} e^{bit} & 0 \\ 0 & e^{-bit} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

$$e^{At} = \frac{1}{\sqrt{2}} \begin{bmatrix} v_1+v_2 & i(v_1-v_2) \end{bmatrix} e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \left(\begin{bmatrix} v_1+v_2 & i(v_1-v_2) \end{bmatrix} \frac{1}{\sqrt{2}} \right)^{-1}$$

real real

$$x(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} v_1+v_2 & i(v_1-v_2) \end{bmatrix} z(t)$$

in z coords

$$z(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} z(0)$$

Purely imaginary eigenvalues $\Longrightarrow$ rotations that don't decay or expand

2 Types of Vector fields

	Conservative	Rotational
linear sys	symmetric A	skew sym A. ←

1\. symmetric matrix

$S = S^T \quad S \in \mathbb{R}^{n \times n}$

hermitian matrix $\quad S \in \mathbb{C}^{n \times n}$

$S = S^*$

Properties

- eigenvalues are real
- eigenvectors are orthogonal

$S \in \mathbb{R}^{n \times n}$ can be diagonalized

$S = RDR^T$

R → orthonormal, D → real

$S \in \mathbb{C}^{n \times n}$

$S = UDU^+$

U → unitary, D → real

" an orthonormal coord. sys that makes a sym. matrix diagonal w/ real eigenvalues"

Quadratic form:

$f: \mathbb{R}^n \to \mathbb{R}$

$f(x) = x^T S x$

↳ probability

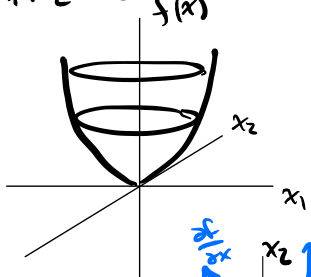
↳ convex opt.

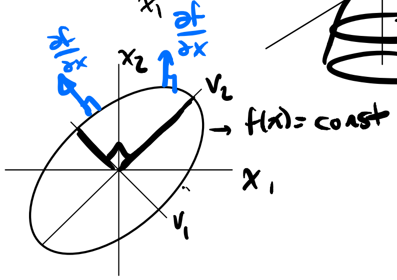
$f(x) = x^T S x$ ← defines a surface

$x \in \mathbb{R}^2 \qquad \lambda_1, \lambda_2 \in \varphi(S)$

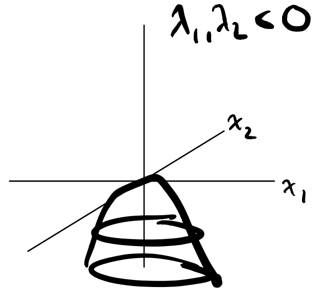
v_1, v_2 evecs of S

$\lambda_1, \lambda_2 > 0$

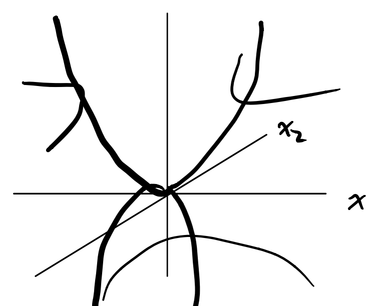


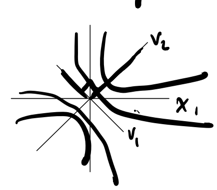


$\lambda_1, \lambda_2 < 0$



$\lambda_1 < 0, \lambda_2 > 0$





$$\frac{\partial f}{\partial x} = x^T(S+S^T)$$

Diff eq. $S = S^T$

$$\dot{x} = Sx \Rightarrow \dot{x} = \frac{\partial f}{\partial x}^T \quad \text{for } f = \frac{1}{2}x^T S x$$

" $\dot{x}$ follows the gradient of some function f "

$\dot{x} = \underset{\sim}{A}x$

if $A \neq A^T$

when does $f(x)$ exist such that $Ax = \frac{\partial f}{\partial x}^T$

general vector field:

$h: \mathbb{R}^n \to \mathbb{R}^n$ (state → derivative)

$$\dot{x} = h(x) = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Conservative vec field cond:

$$\boxed{\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}$$

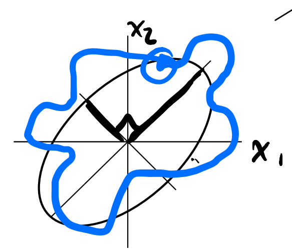
for quadratic forms $x^T S x$ means $S = S^T$

a vector field is conservative if a potential function $f: \mathbb{R}^n \to \mathbb{R}$ exists

↳ energy of the system

potential function surface that you're moving around that defines the energy

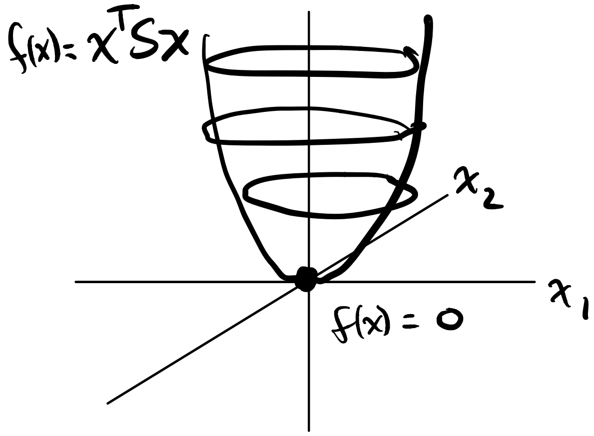
the system flows along gradient of the potential function

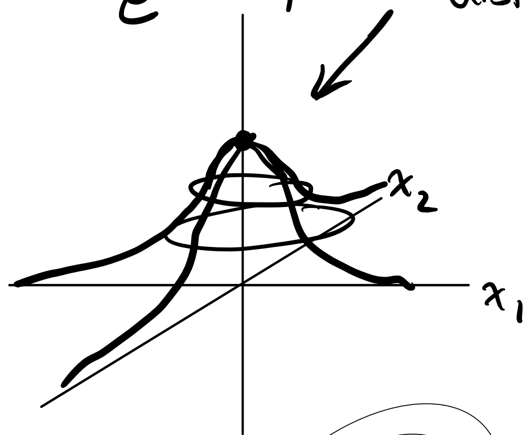


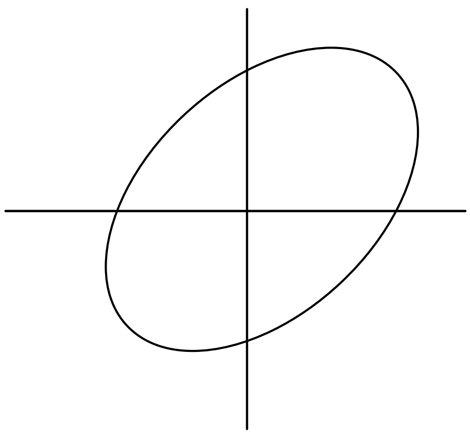
Lagrangian Mechanics

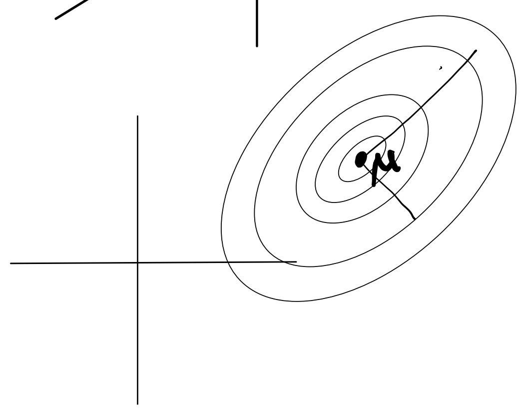
$f(x) = T(x) - V(x) \leftarrow$ inf dim

$e^{-\frac{1}{2}x^TSx}$ Gaussian dist.









Prob. optimization: for a Gaussian dist.

$S = \Sigma^{-1}$

$\ln p(x)$ log likelihood

$-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$

$$p(x) = \frac{1}{(2\pi)^{1/2}\sqrt{\det\Sigma}}\, e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

covariance mean