Topics

- Symmetric / Skew sym
- Helmholtz decomp (linear vec fields)
- complex #'s vs. matrices
- Polar decomposition
- Singular value decomposition

Symmetric Matrices

$S \in \mathbb{R}^{n \times n}$

symmetric: $S = S^T$

$H \in \mathbb{C}^{n \times n}$ $H = H^*$

hermitian

Properties

- real eigenvalues
- orthogonal eigenvectors

Diagonalization

$S = RDR^T \quad S \in \mathbb{R}^{n \times n}$

$R^T R = I \quad \det(R) = 1$

$H = UDU^* \quad H \in \mathbb{C}^{n \times n}$

$U^* U = I \quad \det(U) = 1$

Connection w/ vector fields

$\dot{x} = h(x)$

when is $h(x) = \frac{\partial f}{\partial x}^T$?

"when can you think of $\dot{x}$ as pointing up/down some surface."

Necessary cond: based on 2nd derivative

$$\begin{bmatrix} h_1(x) \\ \vdots \\ h_n(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial h}{\partial x} = \underbrace{\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}}_{\text{symmetric since } \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}}$$

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial x}^T$$

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}$$

Ex. linear case

$\dot{x} = Ax \qquad h(x) = Ax \quad ?$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(Ax) = A \qquad \frac{\partial h}{\partial x} = \frac{\partial h}{\partial x}^T$$

$$A = A^T$$

"linear sys $\bar{w}$ symmetric A

$\Rightarrow$ flow up or down a surface"

potential flow $\quad \Big(\underbrace{f(x)}_{\text{potential function}} = \frac{1}{2} x^T A x \Big)$

Skew Symmetric Matrices

$K \in \mathbb{R}^{n \times n}$

$K = -K^T$

skew symmetric

$K \in \mathbb{C}^{n \times n}$

$K = -K^*$

skew hermitian

Properties

- purely imaginary eigenvalues
- orthogonal eigenvectors

Diagonalize $\quad$ imaginary

$K = \mathcal{U} D \mathcal{U}^*$

$\mathcal{U}^* \mathcal{U} = I \quad \det(\mathcal{U}) = 1.$

For $K \in \mathbb{R}^{n \times n}$

$K = -K^T$

$$K = \mathcal{U} D \mathcal{U}^* = \mathcal{U} \begin{bmatrix} \omega_1 i & 0 & \\ 0 & -\omega_1 i & \\ & & \ddots \end{bmatrix} \mathcal{U}^* = R \begin{bmatrix} 0 & -\omega_1 & \\ \omega_1 & 0 & \\ & & \ddots \end{bmatrix} R^T$$

$R^T R = I \quad \det(R) = 1$

$\begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \to \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\qquad \mathcal{U}$ and R are related (previous lecture)

For $K \in \mathbb{R}^{n\times n}$ $\quad K = -K^T \Rightarrow K_{ii} = 0$ ←

(For $K \in \mathbb{C}^{n\times n}$ $\quad K = -K^* \Rightarrow \mathrm{Re}(K_{ii}) = 0$)

$$K = \begin{bmatrix} 0 & & K_{1n} \\ & \ddots & \\ -K_{1n} & & 0 \end{bmatrix}$$

<u>Fact</u>

$$x^T(Kx) = \sum_{ij} x_i K_{ij} x_j$$

$$= \sum_{i>j} x_i K_{ij} x_j + \sum_{i} x_i \cancelto{0}{K_{ii}} x_i + \underbrace{\sum_{j>i} x_i K_{ij} x_j}_{\sum_{i>j} x_j K_{ji} x_i}$$

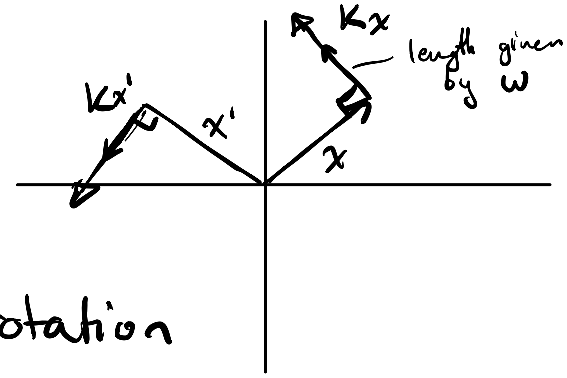
$$= \sum_{i>j} x_i x_j \cancelto{0}{(K_{ij} + K_{ji})}$$

$$x^T K x = 0$$

Kx always $\perp$ to x for any x....

Ex. $K \in \mathbb{R}^{2\times 2}$

$$K = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \leftarrow$$



$\dot{x} = Kx$ is a rotation

solution:

$$x(t) = e^{Kt} x(0)$$

Note: ω_i is rate of rotation

if $K = -K^T$ $\quad K = U\begin{bmatrix} \omega_1 & 0 & \\ 0 & -\omega_1 & \\ & & \ddots \end{bmatrix}U^* \quad R\begin{bmatrix} 0 & -\omega_1 & \\ \omega_1 & 0 & \\ & & \ddots \end{bmatrix}R^T$

$$e^{Kt} = U e^{Dt} U^*$$

$$= U\begin{bmatrix} e^{i\omega_1 t} & 0 & \\ 0 & e^{-i\omega_1 t} & \\ & & \ddots \end{bmatrix}U^*$$

$$= R\begin{bmatrix} \cos(\omega_1 t) & -\sin(\omega_1 t) & \\ \sin(\omega_1 t) & \cos(\omega_1 t) & \\ & & \ddots \end{bmatrix}R^T$$

Summary

$\dot{x} = Kx$ for $K = -K^T$ rotational vector field

i.e. e^{Kt} rotation matrix

Detour (preview of robotics...)

Matrix Group Theory

formalization of idea of symmetries

G : group $\quad G_1, G_2 \in G \Rightarrow G_1 \times G_2 \in G$

$\times$: operation

(G needs to have identity element, ea. element needs to inverse.

Ex. reflections, permutations)

$M = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Leftarrow$ matrix groups

Matrices in these groups are isolated in the vector space of matrices...

Sets of matrices also have a vector interpretation
or a manifold interpretation

you can think about perturbing a matrix

$M + \Delta M \leftarrow$

there are some matrix groups where the elements are <u>not</u> isolated.

ie. for ea. element in the group, there are other elements in that group in any ε- neighborhood in the vector space of matrices

$\Rightarrow$ matrix group that is also a manifold or "a surface"

$\Rightarrow$ <u>Lie groups</u>: groups that are also manifolds

"continuous groups"

Group operations:

$\times$: I, $(\cdot)^{-1}$

$+$: 0, subtraction

$\times, + \rightarrow$ <u>rings</u>

<u>**Galios**</u>

Ex. rotation matrices — $SO(n)$ special orthogonal group in $\mathbb{R}^{n\times n}$

unitary matrices — $SU(n)$ special unitary group in $\mathbb{C}^{n\times n}$

matrices w/ $\det(M) = 1$ — $GL(n)$ general linear group

$\det(M_1 M_2) = \det(M_1)\det(M_2)$

$= 1$

for a element of a Lie group

ex. rotations

there are other elements within a neighborhood

scalar

$$R + \Delta t B \rightarrow \text{could be a rotation}$$

structure on B?

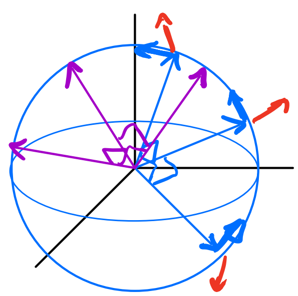
what if B is an element of the group?

$R + \underline{\Delta t}\, R'$ $\quad R, R' \in SO(n)$

$(R^T + \Delta t R'^T)(R + \Delta t R')$

$\underbrace{R^T R}_{I} + \Delta t R'^T R + \Delta t R^T R' + \Delta t^2 \underbrace{R'^T R'}_{I}$

$I(1 + \Delta t^2) + \Delta t R'^T R + \Delta t R^T R'$



$R + \Delta t B$ most B take us off the sphere

$R + \Delta t K$

skew symmetric

skew symmetric matrices are infinitesimal rotations

Lie Group $\longrightarrow$ Lie Algebra

symmetry — instantaneous / infinitesimal symmetry

ex. Rotations — skew symmetric matrices

$R \in SO(n)$ $\quad K \in so(n)$

$R = e^k \xleftarrow{e^{(\cdot)}} K$

"creating a rotation by taking infinitesimal rotating steps

ex. rotations/translations — twists — Robotics

homogeneous transformations

$G \in SE(n) \xleftarrow{e^{(\cdot)}} se(n)$

Back to $\dot{x} = Ax$ ... $A \in \mathbb{R}^{n \times n}$ A not sym or skew sym

$$\dot{x} = Ax = \frac{1}{2}(A + A)x + \frac{1}{2}(A^T - A^T)x$$

$$= \frac{1}{2}\underbrace{(A + A^T)}_{\text{symmetric}}x + \frac{1}{2}\underbrace{(A - A^T)}_{\text{skew symmetric}}x$$

$S = S^T$

$K = -K^T$

A

$(A + A^T)^T = A^T + A$

$(A - A^T)^T = A^T - A = -(A - A^T)$

potential (gradient) vector field + rotational vector field

curl free + divergence free component

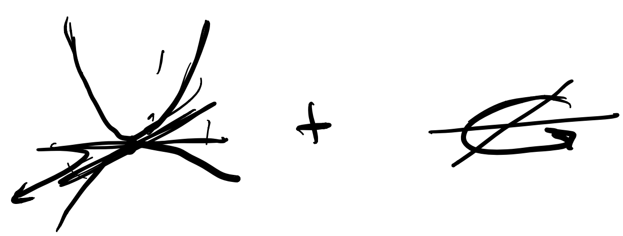
curl free component + divergence free component

special case (linear case) Helmholtz decomposition

3D version

nD version

Note: need to analyze both together to determine stability



any $A \in \mathbb{R}^{n\times n}$

$$A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\substack{\text{average between}\\ \text{a matrix}\\ \text{and its transpose}}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\substack{\text{diff between}\\ \text{a matrix}\\ \text{and its transpose}}}$$

$$= \quad S \quad + \quad K$$

for any $S = S^T$ & $K = -K^T$...

S is "orthogonal" to K...

$\langle S, K \rangle = ?$ $\qquad \langle S, K\rangle = \sum_{ij} S_{ij} K_{ij} = \mathrm{Tr}(S^T K)$

$$\langle S, K\rangle = \sum_{ij} S_{ij} K_{ij} = \sum_{i>j} S_{ij} K_{ij} + \sum_i S_{ii} \cancel{K_{ii}}^{0} + \sum_{j>i} S_{ij} K_{ij}$$

$$= \sum_{i>j} S_{ij} K_{ij} + S_{ji} K_{ji}$$

$$= \sum_{i>j} S_{ij}(\cancel{K_{ij} - K_{ij}}^{0}) = 0$$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

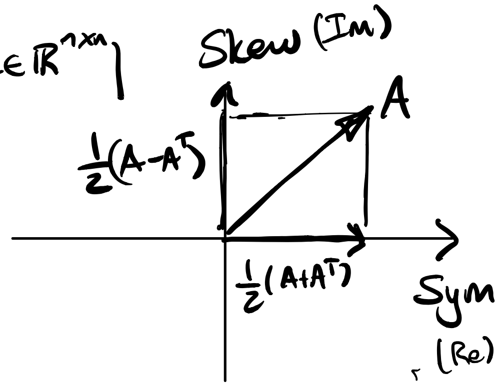
$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$1 + -1$

Summary

$\mathcal{S} = \{ S \mid S = S^T \;\; S \in \mathbb{R}^{n\times n} \}$

$\mathbb{K} = \{ K \mid K = -K^T \;\; K \in \mathbb{R}^{n\times n} \}$



$$\mathbb{R}^{n\times n} = \mathcal{S} \oplus^{\perp} \mathbb{K}$$

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

Analogously - - . $\updownarrow$ $\updownarrow$

$$z = a + bi$$

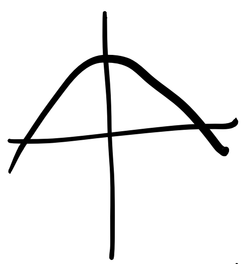
$\sin(\alpha)\cos(\beta)$

$$= \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$

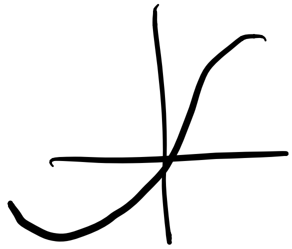
- $e^{i\theta} = \cos\theta + \sin\theta i$
- hyperbolic trig functions

  $\sinh(\theta) = e^{i\theta} - e^{i\theta}$
- odd, even functions

even



odd



$f(x) = f(-x)$

ex $\cos(x) = \cos(-x)$

$f(x) = - f(-x)$

$\sin(x) = -\sin(-x)$

preview $A = PR$ polar decomposition

Side Note:

Matrix norms

Vector . . .

Frobenius

$\|A\|_F = \left(\sum_{ij} A_{ij}^2\right)^{1/2}$

"vector 2-norm"

$\|A\|_F = \sqrt{\langle A, A\rangle}$

$= \sqrt{Tr(A^T A)}$"

→ "vector 1-norm

operator norms

induced norms

$\uparrow$

$\uparrow$

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

$re^{i\theta} = r\cos\theta + r\sin\theta i$

even func

odd func.

preview homogeneous

$$G = SE(3)$$

rotations & translations in $\mathbb{R}^3$

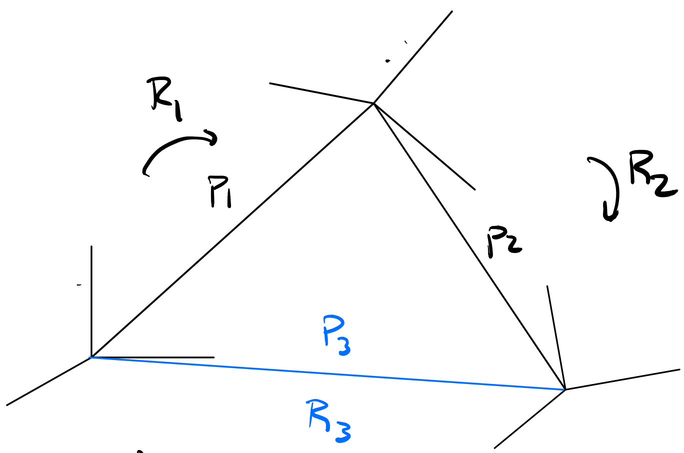
rotation in $\mathbb{R}^3$ translation by $p \in \mathbb{R}^3$

$$G = \begin{bmatrix} R & p \\ 000 & 1 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} R_1 & P_1 \\ 000 & 1 \end{bmatrix} \quad G_2 = \begin{bmatrix} R_2 & P_2 \\ 000 & 1 \end{bmatrix}$$

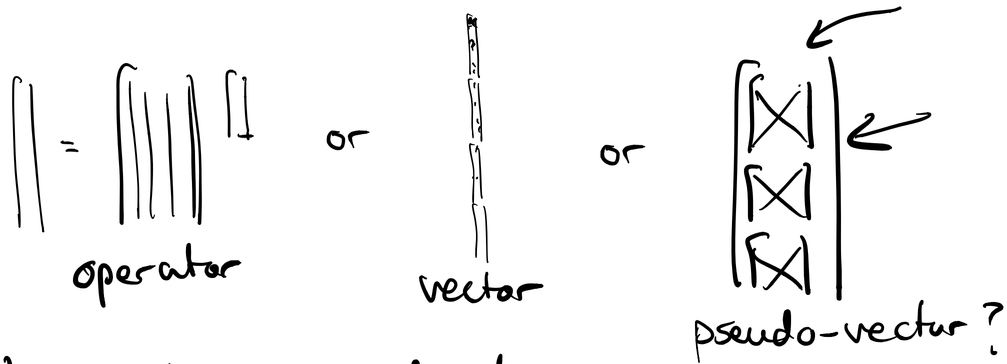
rotation translation

$$G_1 G_2 = \begin{bmatrix} R_1 R_2 & R_1 P_2 + P_1 \\ \underline{0\,0\,0} & \underline{1} \end{bmatrix}$$



$$\begin{bmatrix} R_3 & P_3 \\ 000 & 1 \end{bmatrix} = \begin{bmatrix} R_1 & P_1 \\ 000 & 1 \end{bmatrix} \begin{bmatrix} R_2 & P_2 \\ 000 & 1 \end{bmatrix}$$

$$R_3 = R_1 R_2 \qquad P_3 = R_1 P_2 + P_1$$



kronecker products...