$V = [v_1 \cdots v_n]$ $\quad$ v_i eigenvector of A^TA

symmetric

→ real eigenvalues

→ real eigenvectors (orthonormal)

$U = [u_1 \cdots u_m]$ $\quad$ $u_i = \dfrac{A v_i}{\sigma_i}$

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

ROBOTICS: Robotic Manipulation Murray, Li, Sastry

Controlling:

- Manipulation (rigid body transforms, kinematics motion dynamics
  Linear algebra heavy
  quaternions)
- Motion planning (search algorithms
  prob. search alg.)
- Sensing (lidar, estimation, filtering)
  computer vision] hard

Robotic Manipulation:

Vectors $x \in \mathbb{R}^3$

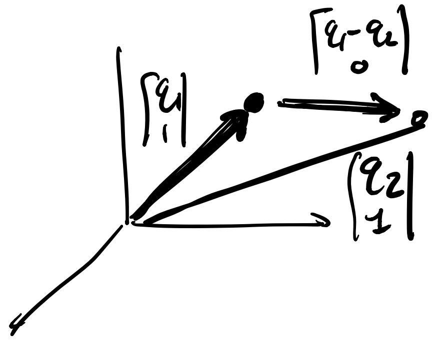
$\rightarrow \begin{bmatrix} q \\ 1 \end{bmatrix}$ ← point in space

$q \in \mathbb{R}^3$: position vector

Difference between vectors

$$\begin{bmatrix} q_1 \\ 1 \end{bmatrix} - \begin{bmatrix} q_2 \\ 1 \end{bmatrix} = \begin{bmatrix} q_1 - q_2 \\ 0 \end{bmatrix}$$

direction



Lie Groups

Lie Algebra

"generate the lie group"

skew symmetric

Rotations in $\mathbb{R}^3$

$$SO(3) = \{ R \mid R^T R = I, \det(R) = 1, R \in \mathbb{R}^{3\times 3} \}$$

$$so(3) = \{ k \mid k = -k^T, k \in \mathbb{R}^{3\times 3} \}$$

$$\boxed{R = e^{kt}}$$

$$k = U \begin{bmatrix} bi & 0 & 0 \\ 0 & -bi & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*$$

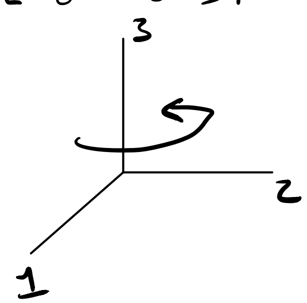
$$R = e^{kt} = \underline{U} \underline{\begin{bmatrix} e^{bit} & 0 & 0 \\ 0 & e^{-bit} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underline{U^*}$$

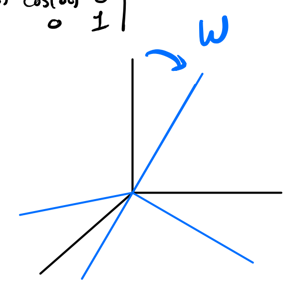
$$R = U \underbrace{\begin{bmatrix} e^{bit} & 0 & 0 \\ 0 & e^{-bit} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{} U^*$$

$$R = \underbrace{U \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{W \leftarrow \text{rotation}} \underbrace{\begin{bmatrix} \cos(bt) & -\sin(bt) & 0 \\ \sin(bt) & \cos(bt) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\uparrow\ \text{rotation}} \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^* U^*}_{W^T \leftarrow \text{rotation}}$$

$$R = W \begin{bmatrix} \cos(bt) & -\sin(bt) & 0 \\ \sin(bt) & \cos(bt) & 0 \\ 0 & 0 & 1 \end{bmatrix} W^T$$

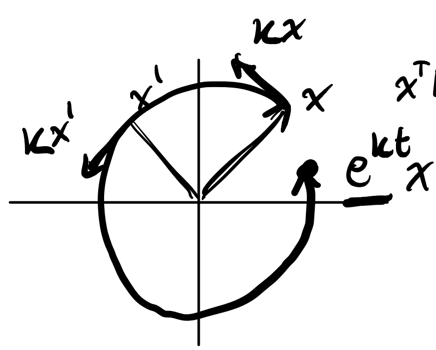
$$\begin{bmatrix} \cos(bt) & -\sin(bt) & 0 \\ \sin(bt) & \cos(bt) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = W \begin{bmatrix} \cos(bt) & -\sin(bt) & 0 \\ \sin(bt) & \cos(bt) & 0 \\ 0 & 0 & 1 \end{bmatrix} W^T \leftarrow$$



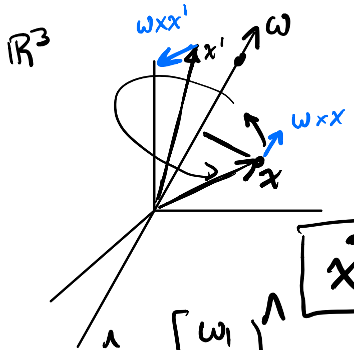


Linear vector field:

$$\dot{x} = Kx \longrightarrow x(t) = \underline{e}^{kt} x(0)$$



$x^T K x = 0$



axis of rotation $\omega \in \mathbb{R}^3$

$$\dot{x} = \omega \times x$$

how to turn into a skew symmetric matrix

$$\boxed{\dot{x} = \hat{\omega} x}$$

$$\hat{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \leftarrow \text{skew symmetric}$$

$\omega \times x = \hat{\omega} x \leftarrow$ you can check this

$$x^T \hat{\omega} x = 0$$

what if $x = a\omega$?

$$\dot{x} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} a$$

points on the axis are fixed $\rightarrow$ $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\hat{\omega}\omega = 0$$

$$\dot{x} = b\hat{\omega} x$$

$$x(t) = e^{\hat{\omega}(bt)} x(0) \implies$$

direction of ω tells you the axis of rotation

magnitude of ω tells you the rate of rotation

Computing a rotation matrix:

axis, rate $\rightarrow$ ω $\quad \frac{\omega}{|\omega|} =$ axis

$|\omega| =$ rate

$$R = e^{\hat{\omega} t}$$
$$= I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2} + \frac{(\hat{\omega} t)^3}{3!} + \cdots$$
$$= I + \hat{\omega}\sin(t) + \hat{\omega}^2(1 - \cos(t))$$

Rodriguez formula

Euler Angles:

$$R = R_x(\phi) R_y(\beta) R_z(\alpha)$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \quad R_y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \quad R_z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question: for a particular R.
are ϕ, β, α unique? $\longrightarrow$ <u>NO</u>

Beyond Rotations to rigid transformations

$q \in \mathbb{R}^3 \qquad \bar{g}(q) = \underset{\text{rotation}}{R} q + \underset{\text{translation}}{P}$

Lie Group

$$SE(3) = \underbrace{\mathbb{R}^3}_{\text{translation vectors}} \times \underbrace{SO(3)}_{\text{rotations}}$$

homogeneous or "rigid" transformations in $\mathbb{R}^3$

Matrix form:

$$g = \begin{bmatrix} R & P \\ 000 & 1 \end{bmatrix} \qquad R \in \mathbb{R}^{3\times 3}, \; P \in \mathbb{R}^3$$

$$\bar{g}(q) = g \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} R & P \\ 000 & 1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} Rq + P \\ 1 \end{bmatrix}$$

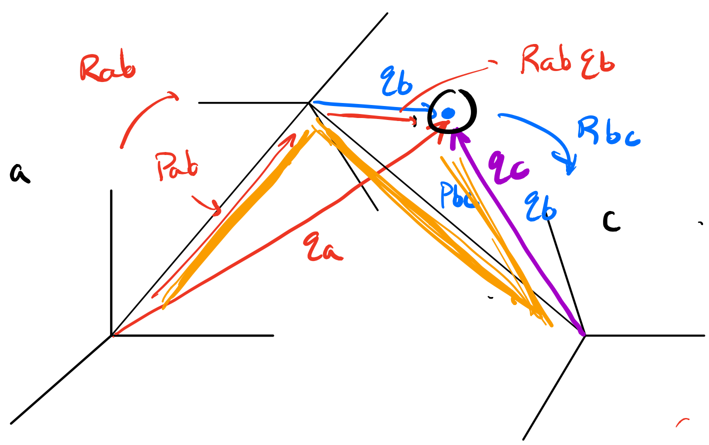
$$g_1 = \begin{bmatrix} R_1 & P_1 \\ 000 & 1 \end{bmatrix} \quad g_2 = \begin{bmatrix} R_2 & P_2 \\ 000 & 1 \end{bmatrix} \quad g_1 g_2 = \begin{bmatrix} R_1 R_2 & R_1 P_2 + P_1 \\ 000 & 1 \end{bmatrix}$$

rotation translation

$$\begin{bmatrix} q' \\ 1 \end{bmatrix} = g_1 g_2 \begin{bmatrix} q \\ 1 \end{bmatrix}$$

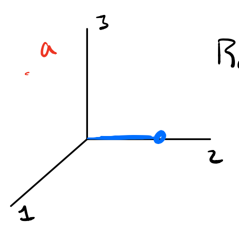
$$g_2 g_1 = \begin{bmatrix} R_2 R_1 & R_2 P_1 + P_2 \\ 000 & 1 \end{bmatrix}$$

Picture

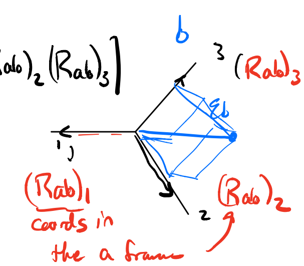


$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & P_{ab} \\ 000 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} q_b + P_{ab} \\ 1 \end{bmatrix}$$

Matrix forms of rigid allow us to do coord transforms w/ matrix multiplication



$$R_{ab} = \begin{bmatrix} (R_{ab})_1 & (R_{ab})_2 & (R_{ab})_3 \end{bmatrix}$$



$$q_a = \begin{bmatrix} R_{ab} \end{bmatrix} q_b$$

$$g_{ab} = \begin{bmatrix} R_{ab} & P_{ab} \\ 000 & 1 \end{bmatrix}$$

$$g_{bc} = \begin{bmatrix} R_{bc} & P_{bc} \\ 000 & 1 \end{bmatrix}$$

$$\begin{bmatrix} q_b \\ 1 \end{bmatrix} = \begin{bmatrix} R_{bc} & P_{bc} \\ 000 & 1 \end{bmatrix} \begin{bmatrix} q_c \\ 1 \end{bmatrix} = \begin{bmatrix} R_{bc} q_c + P_{bc} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & P_{ab} \\ 000 & 1 \end{bmatrix} \begin{bmatrix} R_{bc} & P_{bc} \\ 000 & 1 \end{bmatrix} \begin{bmatrix} q_c \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} R_{bc} q_c + R_{ab} P_{bc} + P_{ab} \\ 1 \end{bmatrix}$$

Inverse Transforms

$$R_{ba} = (R_{ab})^{-1} = R_{ab}^T \qquad -P_{ab}$$

$$\boxed{g_{ba} = (g_{ab})^{-1} = \begin{bmatrix} R_{ab}^T & -R_{ab}^T P_{ab} \\ 000 & 1 \end{bmatrix}} = \begin{bmatrix} R_{ba} & -R_{ba}P_{ab} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ba} & P_{ba} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_{ab} & P_{ab} \\ 000 & 1 \end{bmatrix}^{-1} = \left[\begin{bmatrix} R_{ab} & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} I & R_{ab}^T P_{ab} \\ 0 & 1 \end{pmatrix}\right]^{-1}$$

$$= \begin{bmatrix} I & -R_{ab}^T P_{ab} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_{ab}^T & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{ab}^T & -R_{ab}^T P_{ab} \\ 0 & 1 \end{bmatrix}$$

Detour

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

skew sym $\hat{\omega}$ $\longrightarrow$ rotation $R = e^{\hat{\omega} t}$

is there an equiv for homogeneous transforms?

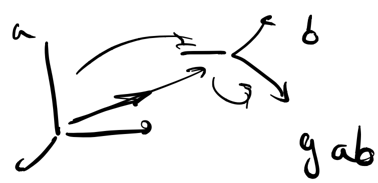
ω : rotation axis & rate

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \begin{matrix} \leftarrow \text{velocity} \\ \leftarrow \text{rotation axis \& rate} \end{matrix}$$

$\underline{\xi \in \mathbb{R}^6} \leftarrow \underline{\text{twist}}$

rotation & translation

$$\hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0\,0\,0 & 0 \end{array}\right] \in \mathbb{R}^{4\times 4}$$



$$g = e^{\hat{\xi} t} \Big\} \rightarrow \text{homogeneous transformation}$$

$$= I + \hat{\xi} t + \frac{(\hat{\xi} t)^2}{2} + \frac{(\hat{\xi} t)^3}{3!} + \cdots$$

if $\omega \neq 0$

$$e^{\hat{\xi} t} = \begin{bmatrix} e^{\hat{\omega} t} & (I - e^{\hat{\omega} t})\hat{\omega} v + \omega\omega^T v t \\ 0\,0\,0 & 1 \end{bmatrix}$$

Equivalent of Rodriguez formula (ish)

if $\omega = 0$

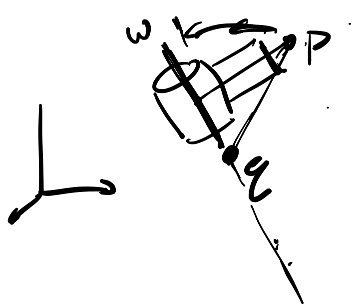
$$e^{\hat{\xi} t} = \begin{bmatrix} I & vt \\ 0\,0\,0 & 1 \end{bmatrix}$$

Transformations in Robotics:

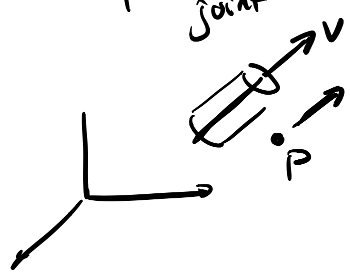
Robotic manipulator: joints.

2 types

Rotational joint



prismatic (linear) joint



Rotational joint:

$$\dot{p} = \underline{\hat{\omega}}(p-q) = \hat{\omega}p - \hat{\omega}q$$

$$\begin{bmatrix} \dot{p} \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{\xi} \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q \\ 000 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q \\ 000 & 0 \end{bmatrix}$$

Prismatic joint:

$$\dot{p} = v$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 000 & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 000 & 0 \end{bmatrix}$$

Total motion of ea joint

$$\boxed{\begin{aligned} e^{\hat{\xi}t} &= \begin{bmatrix} e^{\hat{\omega}t} & (I-e^{\hat{\omega}t})\hat{\omega}(-\hat{\omega}q) - \omega\overset{0}{\cancel{\omega^T\hat{\omega}}}q t \\ 000 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\hat{\omega}t} & -(I-e^{\hat{\omega}t})\hat{\omega}^2 q \\ 000 & 1 \end{bmatrix} \end{aligned}}$$

$$\boxed{e^{\hat{\xi}t} = \begin{bmatrix} I & vt \\ 000 & 1 \end{bmatrix}}$$

if $\omega \neq 0$

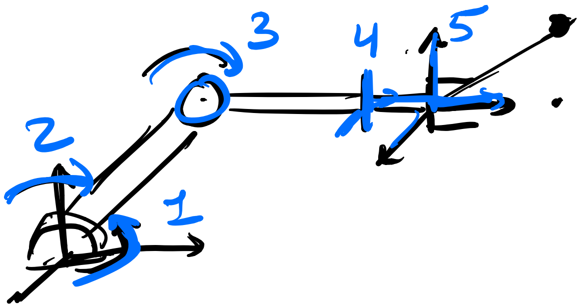
$$e^{\hat{\xi}t} = \begin{bmatrix} e^{\hat{\omega}t} & (I-e^{\hat{\omega}t})\hat{\omega}v + \omega\omega^T v t \\ 000 & 1 \end{bmatrix}$$

Equivalent of Rodriguez formula (ish)

if $\omega = 0$

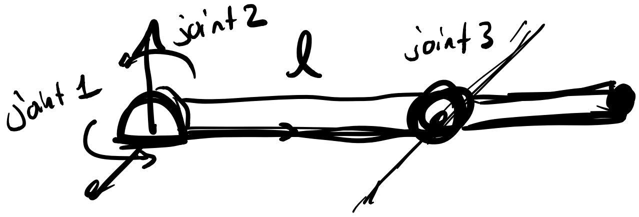
$$e^{\hat{\xi}t} = \begin{bmatrix} I & vt \\ 000 & 1 \end{bmatrix}$$

FORWARD KINEMATICS



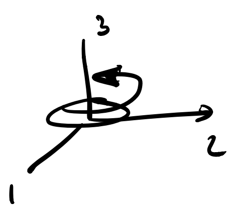
how do convert backwards & forwards between the base coord frame & the end effector.

1. Layout arm in an initial simple configuration



2. write twists of joints in initial configuration

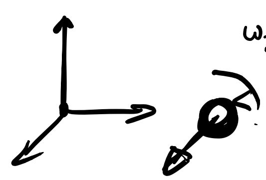
joint 1



$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad q_1 = 0$$

$$\hat{\xi}_1 = \begin{bmatrix} \hat{\omega}_1 & -\hat{\omega}_1 q_1 \\ 000 & 0 \end{bmatrix}$$

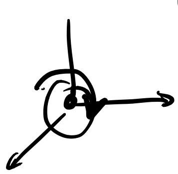
joint 3



$$\omega_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}$$

$$\hat{\xi}_3 = \begin{bmatrix} \hat{\omega}_3 & -\hat{\omega}_3 q_3 \\ 000 & 0 \end{bmatrix}$$

joint 2



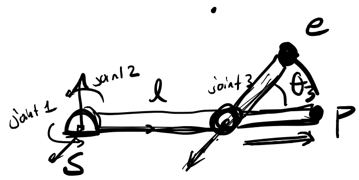
$$\omega_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = 0$$

$$\hat{\xi}_2 = \begin{bmatrix} \hat{\omega}_2 & -\hat{\omega}_2 q_2 \\ 000 & 0 \end{bmatrix}$$

3. compute homogeneous transforms

$$g_1 = e^{\hat{\xi}_1\theta_1} \quad g_2 = e^{\hat{\xi}_2\theta_2} \quad g_3 = e^{\hat{\xi}_3\theta_3}$$

4. compute product of exponentials formula

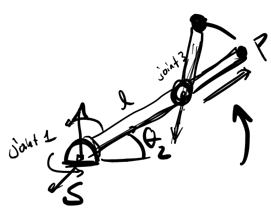


Forward kinematics

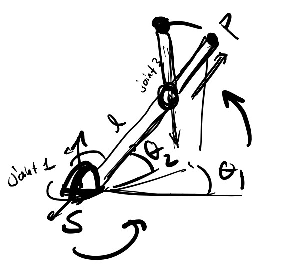
$$\begin{bmatrix}p_s\\1\end{bmatrix} = \underset{e}{g_3}\begin{bmatrix}p_e\\1\end{bmatrix} = e^{\hat{\xi}_3\theta_3}\begin{bmatrix}p_e\\1\end{bmatrix}$$

$$\ln(g) = \hat{\xi}_1\theta_1 + \hat{\xi}_2\theta_2 + \hat{\xi}_3\theta_3$$

??



$$\begin{bmatrix}p_s\\1\end{bmatrix} = g_2 g_3\begin{bmatrix}p_e\\1\end{bmatrix} = e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}\begin{bmatrix}p_e\\1\end{bmatrix}$$



$$\begin{bmatrix}p_s\\1\end{bmatrix} = g_1 g_2 g_3\begin{bmatrix}p_e\\1\end{bmatrix} = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}\begin{bmatrix}p_e\\1\end{bmatrix}$$

nonlinear

$$g_{se} = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} \qquad g(\theta_1, \theta_2, \theta_3)$$

Inverse kinematics: (harder) → needs to be broken down into manageable pieces

given g → compute θ_i's

Paden-Khan subproblems

$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} e^{\hat{\omega}_1\theta_1} & 0 \\ 0 & 1 \end{bmatrix} \qquad e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} e^{\hat{\omega}_2\theta_2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\omega}_1\theta_1} e^{\hat{\omega}_2\theta_2} = e^{\hat{\omega}_2\theta_2} e^{\hat{\omega}_1\theta_1} \ ?$$

Depends on joint construction

$$\begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_2 & -s\theta_2 \\ 0 & s\theta_2 & c\theta_2 \end{bmatrix} \leftarrow$$

- gantry - - -
  sliding.
- ball & socket.