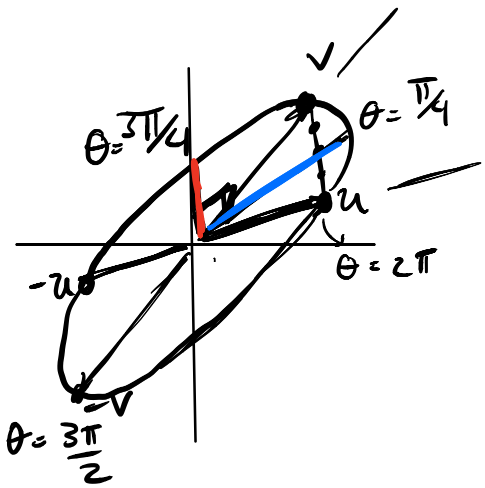
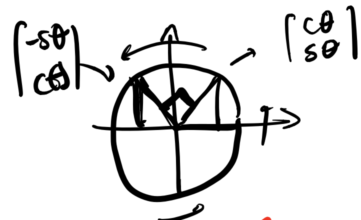
Shape of orbits $\leftarrow$ complex eigenvalues



$$\begin{bmatrix} u & v \end{bmatrix} \boxed{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}$$

$\cos\theta\, u + \sin\theta\, v$



$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \longrightarrow \begin{bmatrix} u & v \end{bmatrix} \boxed{\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}}$$

$$\begin{bmatrix} c\theta & s\theta \end{bmatrix} \begin{bmatrix} -s\theta \\ c\theta \end{bmatrix} = 0$$

complex form $\quad \lambda, \lambda^* = e^{\pm i\theta}$

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1} \longrightarrow \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} c\theta & s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix}^{-1} \qquad p_1 = p_2^*$$

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} w \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} w \end{bmatrix}^{-1} \begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & -i \\ i & i \end{bmatrix}^{-1} \frac{1}{\sqrt{2}} \qquad \left[\begin{bmatrix} p_1 & p_2 \end{bmatrix} w\right]^{-1}$$

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \begin{bmatrix} p_1 + p_2 & i(p_1 - p_2) \end{bmatrix}$$

real

$p_1 = (u + vi)\frac{1}{\sqrt{2}}$

$p_2 = (u - vi)\frac{1}{\sqrt{2}}$

$\begin{bmatrix} p_1 & p_2 \end{bmatrix} u = \begin{bmatrix} u & v \end{bmatrix}$

$$A = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1} = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix}^{-1}$$

$$Ax \qquad x = \begin{bmatrix} u & v \end{bmatrix} z \qquad Ax = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{matrix} z_1 \\ z_2 \end{matrix}$$

Circulant Matrices $\Longrightarrow$ Discrete Fourier Transform DFT

1. Shift Matrix

$$S = \begin{bmatrix} 0 & 0 & & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad S = \mathbb{R}^{n\times n}$$

$$S = [s_1 \cdots s_n]$$

$$SS = [Ss_1 \cdots Ss_n]$$

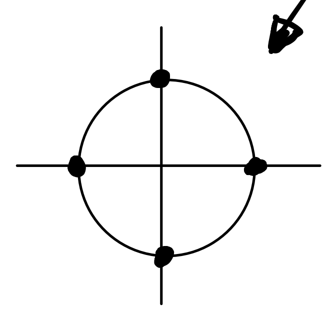
$$\begin{bmatrix} x_n \\ x_1 \\ \vdots \\ x_2 \end{bmatrix} = S \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & & 1 & 0 \\ 0 & & & 1 \\ 1 & & & \\ \vdots & \ddots & & \\ 0 & & 1\,0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What are eigenvectors & eigenvalues of S?

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbb{1} \qquad \lambda = 1$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ i \\ -1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \qquad \lambda = -i$$



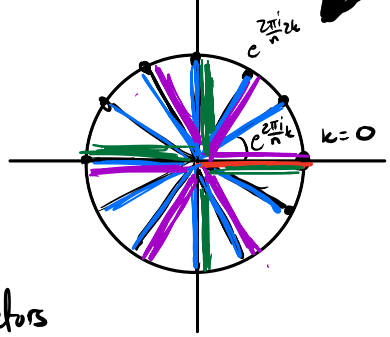
k-th eigenvector: $F_k \in \mathbb{C}^n$

$(F_k)_j = e^{\frac{2\pi i jk}{n}}$

$$F_k = \begin{array}{c} \uparrow \\ j \\ \downarrow \end{array} \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{n}k} \\ e^{\frac{2\pi i}{n}2k} \\ \vdots \\ e^{\frac{2\pi i (n-1)k}{n}} \end{bmatrix} \begin{array}{l} \leftarrow j=0 \\ \leftarrow j=1 \\ \leftarrow j=2 \\ \\ \leftarrow j=n-1 \end{array}$$

$k=0 \qquad F_k = \mathbb{1}$

F_k:



k: "frequency of oscillation"

F_k: harmonic shape

eigenvectors of S

F_k : $k = 0, 1, 2, 3, \ldots, n-1$

$S \in \mathbb{R}^{n\times n}$

$$SF_k = \begin{bmatrix} e^{\frac{2\pi i (n-1)k}{n}} \\ 1 \\ e^{\frac{2\pi i k}{n}} \\ \vdots \\ e^{\frac{2\pi i (n-2)k}{n}} \end{bmatrix} = \lambda F_k = \lambda \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{n}k} \\ \vdots \end{bmatrix} \qquad \lambda_k = e^{-\frac{2\pi i}{n}k}$$

$S = FDF^{-1} \qquad F \in \mathbb{C}^{n\times n}$

$$[F]_{jk} = e^{\frac{2\pi i jk}{n}}$$

row col

tot dim

$j = 0, 1, \ldots, n-1$

$k = 0, 1, \ldots, n-1$

$$F = \begin{array}{c} \uparrow \\ j \\ \downarrow \end{array} \overset{\longleftarrow\ k\ \longrightarrow}{\begin{bmatrix} 1 & & & & 1 \\ \vdots & e^{\frac{2\pi i}{n}} & e^{\frac{2\pi i}{n}2} & \cdots & e^{\frac{2\pi i}{n}(n-1)} \\ \vdots & e^{\frac{2\pi i 2}{n}} & e^{\frac{2\pi i 2\times 2}{n}} & \ddots & \\ 1 & e^{\frac{2\pi i (n-1)}{n}} & & & e^{\frac{2\pi i}{n}(n-1)(n-1)} \end{bmatrix}}$$

← cols are right eigen vectors for S.

cols of F are the discrete fourier basis vectors

F_k : sinusoidal signals in discrete time

k: frequency index

j: time index.

mathematically can't tell one from the other

$\frac{1}{\sqrt{n}} F$ is unitary

$$F_k^* F_k = 1 \cdot 1 + \underbrace{e^{\frac{-2\pi i k}{n}} e^{\frac{2\pi i k}{n}}}_{1} + \underbrace{e^{\frac{-2\pi i 2k}{n}} e^{\frac{2\pi i 2k}{n}}}_{1} + \cdots = n$$

$$F_{k'}^* F_k = 0 \quad \text{if} \quad k' \neq k$$

$$S = \frac{1}{\sqrt{n}} F D F^* \frac{1}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} F_0 & \cdots & F_{n-1} \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & e^{\frac{-2\pi i}{n}} & & \\ & & \ddots & \\ 0 & & & e^{\frac{-2\pi i (n-1)}{n}} \end{bmatrix} \begin{bmatrix} - F_0^* - \\ \vdots \\ - F_{n-1}^* - \end{bmatrix} \frac{1}{\sqrt{n}}$$

$$\lambda_k = e^{\frac{-2\pi i k}{n}}$$

F_k : discrete time sinusoid.

DISCRETE TIME: $(F_k)_j = e^{\frac{2\pi i j k}{n}}$

CONT TIME: $g(\omega, t) = e^{(i\omega t)}$

$= \cos(\omega t) + i \sin(\omega t)$

$j \leftrightarrow t$

$\frac{2\pi k}{n} \leftrightarrow \omega$

$$x = Fz \qquad z = \begin{bmatrix} z_0 \\ \vdots \\ z_{n-1} \end{bmatrix}$$

$$= F_0 z_0 + F_1 z_1 + \dots + F_{n-1} z_{n-1}$$

time signal

z : "coeffs you multiply by sinusoids to get the time signal x"

$$z = F^{-1} x$$

$$= \frac{1}{n} F^* x \implies \underbrace{z = \frac{1}{n} F^* x}_{\text{DFT}}$$

DISCRETE FOURIER TRANSFORM

Note: Fourier transform in continuous time is similar — inf dim change of basis

F_k : eigenvector of S ← shift matrix. fin

e^{wt} : eigenfunction of $\frac{d}{dt}$ ← derivative operator inf

Circulant Matrix:

$c \in \mathbb{C}^n$ $\quad c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$

$$C = \begin{bmatrix} c_0 & c_{n-1} & & c_1 \\ c_1 & c_0 & & \\ \vdots & c_1 & & \\ \vdots & \vdots & & c_{n-1} \\ c_{n-1} & c_{n-2} & c_1 & c_0 \end{bmatrix}$$

every time shifted version of c. thinking of c as a periodic signal

$C^T x$: discrete convolution of x & c

$$c * x = \begin{bmatrix} c_0 & c_{n-1} & & c_1 \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix} =$$

$$S = \begin{bmatrix} 0 & & 1 \\ 1 & \ddots & \\ & & 1 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & & 1 \\ 1 & \ddots & \end{bmatrix}$$

$$C = c_0 I + c_1 S + c_2 S^2 \cdots c_{n-1} S^{n-1}$$

$$S = \frac{1}{\sqrt{n}} F D F^* \frac{1}{\sqrt{n}}$$

spectral mapping thm

$$C = \frac{1}{\sqrt{n}} F \left[c_0 I + c_1 D + c_2 D^2 \cdots + c_{n-1} D^{n-1} \right] F^* \frac{1}{\sqrt{n}}$$

↳ any circulant matrix can be diagonalized by the DFT matrix.

"any periodic signal can be written as a finite linear comb of sines & cosines"

$$C = \frac{1}{\sqrt{n}} F \left[c_0 I + c_1 D + c_2 D^2 \cdots + c_{n-1} D^{n-1} \right] F^* \frac{1}{\sqrt{n}}$$

$$D = \begin{bmatrix} 1 & & 0 \\ & e^{\frac{-2\pi i}{n}} & \\ 0 & \ddots & e^{\frac{-2\pi i(n-1)}{n}} \end{bmatrix} \quad D^2 = \begin{bmatrix} 1 & & \\ & e^{\frac{-2\pi i 2}{n}} & \\ & & e^{\frac{-2\pi i 2(n-1)}{n}} \end{bmatrix} \quad \text{etc.}$$

$$C = \frac{1}{\sqrt{n}} F \left[\operatorname{diag}(F^* c) \right] F^* \frac{1}{\sqrt{n}}$$

matrix w vector on the diagonal.

$C^T x$: convolution

messy.

watchout for $\frac{1}{n}$'s floating around.

$$C^T x = F (\operatorname{dg} F^* c)^* \frac{1}{n} \underbrace{F^* x}_{z}$$

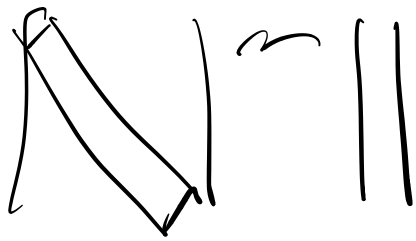
$$\frac{1}{n} F^* (C^T x) = (\operatorname{dg} F^* c)^* z$$

DFT of x.

DFT of c

elementwise multiplication of $F^* c$ & z

convolution in the time domain is multiplication in the frequency domain.



$$dg(z')z$$

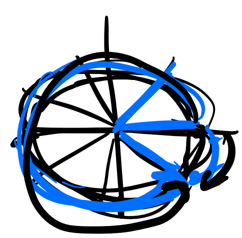
$$\begin{bmatrix} z_1' & & 0 \\ & \ddots & \\ 0 & & z_n' \end{bmatrix}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_1' z_1 \\ \vdots \\ z_n' z_n \end{bmatrix}$$

Conjugate pairs of DFT vectors:

$$F = \begin{array}{c} \\ j \downarrow \end{array} \begin{array}{c} \xleftarrow{\quad k \quad}\!\!\!\longrightarrow \\ \begin{bmatrix} 1 & e^{\frac{2\pi i}{n}} & e^{\frac{2\pi i}{n}2} & \cdots & e^{\frac{2\pi i}{n}(n-2)} & e^{\frac{2\pi i}{n}(n-1)} \\ \vdots & e^{\frac{2\pi i 2}{n}} & e^{\frac{2\pi i 2\times 2}{n}} & \ddots & & \\ & \vdots & & & \ddots & \\ 1 & e^{\frac{2\pi i(n-1)}{n}} & & & e^{\frac{2\pi i(n-2)(n-1)}{n}} & e^{\frac{2\pi i}{n}(n-1)(n-1)} \end{bmatrix} \end{array}$$

$$e^{\frac{2\pi i}{n}(n-1)} = e^{\frac{2\pi i}{n}(-1)}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$



$$F = \begin{array}{c} \\ j \downarrow \end{array} \begin{array}{c} \xleftarrow{\quad k \quad}\!\!\!\longrightarrow \\ \begin{bmatrix} 1 & e^{\frac{2\pi i}{n}} & e^{\frac{2\pi i}{n}2} & \cdots & e^{\frac{2\pi i}{n}(-2)} & e^{\frac{2\pi i}{n}(-1)} \\ \vdots & e^{\frac{2\pi i 2}{n}} & e^{\frac{2\pi i 2\times 2}{n}} & \ddots & & \\ & \vdots & & & \ddots & \\ 1 & e^{\frac{2\pi i(n-1)}{n}} & & & e^{\frac{2\pi i(-1)(-2)}{n}} & e^{\frac{2\pi i}{n}(-1)(-1)} \end{bmatrix} \end{array}$$

$$F = \begin{bmatrix} F_0 & F_1 & \cdots & F_{n-1} \end{bmatrix} \qquad F = \begin{bmatrix} F_0 & F_1 & F_2 & \cdots & F_{-2} & F_{-1} \end{bmatrix}$$

$$F = \begin{bmatrix} \underline{F_0 \cdots F_k \cdots} & \boxed{\;} & \underline{F_{-k} \cdots} \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \boxed{+i\,+i} \quad \boxed{+i\,-i} \\ 1 & \boxed{-i\,+i} \quad \boxed{-i\,-i} \end{bmatrix} \qquad F^* =$$

<u>FAST</u> <u>FOURIER</u> <u>TRANSFORM</u>

$$F_{jk} = e^{\frac{2\pi i jk}{n}}$$

$$k - k' =$$

$$F_{k'}^* F_k = \sum_j \underline{e^{\frac{2\pi i j}{n}(k-k')}} \Rightarrow 0$$

$$s^n = 1 \Rightarrow s^n - 1 = 0$$

$$(s - z_1)(s - z_2)(s - z_3) \cdots = 0$$

$$s^n + (\underline{z_1 + z_2 + \cdots z_n}) s^{n-1} + \cdots \quad 1 = 0$$

S^n_+ —— -1