BLOCK MATRIX MULTI.

$$AB = A\begin{bmatrix} B_1 \cdots B_p \end{bmatrix} = \begin{bmatrix} AB_1 \cdots AB_p \end{bmatrix}$$

2 MORE EXAMPLES:

① $ABC = \begin{bmatrix} - \bar{a}_1^T - \\ - \bar{a}_m^T - \end{bmatrix} \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} | & & | \\ c_1 & \cdots & c_q \\ | & & | \end{bmatrix}$

$A \in \mathbb{R}^{m \times n}$

$B \in \mathbb{R}^{n \times p}$

$C \in \mathbb{R}^{p \times q}$

$$\begin{bmatrix} - \bar{a}_1^T - \\ - \bar{a}_m^T - \end{bmatrix} \begin{bmatrix} Bc_1 \cdots Bc_q \end{bmatrix}$$

$$\begin{bmatrix} \bar{a}_1^T \boxed{B} c_1 & \cdots & \bar{a}_1^T \boxed{B} c_q \\ \bar{a}_m^T \boxed{B} c_1 & \cdots & \bar{a}_m^T \boxed{B} c_q \end{bmatrix}$$

Recall...

$$\begin{bmatrix} - \bar{a}_1^T - \\ - \bar{a}_m^T - \end{bmatrix} \begin{bmatrix} | & & | \\ c_1 & \cdots & c_q \\ | & & | \end{bmatrix} = \begin{bmatrix} \bar{a}_1^T c_1 & \bar{a}_1^T c_q \\ \bar{a}_m^T c_1 & \bar{a}_m^T c_q \end{bmatrix}$$

② $ABC = \begin{bmatrix} A_1 | \cdots | A_n \end{bmatrix} \begin{bmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & & \vdots \\ B_{n1} & \cdots & B_{np} \end{bmatrix} \begin{bmatrix} - \bar{c}_1^T - \\ \vdots \\ - \bar{c}_p^T - \end{bmatrix}$

$$= \begin{bmatrix} A_1B_{11} + \cdots + A_nB_{n1} \,\Big|\, \cdots \,\Big|\, A_1B_{1p} + \cdots + A_nB_{np} \end{bmatrix} \begin{bmatrix} - \bar{c}_1^T - \\ \vdots \\ - \bar{c}_p^T - \end{bmatrix}$$

$$= \left(A_1B_{11}\bar{c}_1^T + \cdots + A_nB_{n1}\bar{c}_1^T \right) + \cdots + \left(A_1B_{1p}\bar{c}_p^T + \cdots + A_nB_{np}\bar{c}_p^T \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{p} \left(A_i B_{ij} \bar{c}_j^T \right) = \sum_{ij} A_i B_{ij} \bar{c}_j^T$$

$$\begin{bmatrix} A_i \end{bmatrix} B_{ij} \begin{bmatrix} - \bar{c}_j^T - \end{bmatrix}$$

$$\overset{3}{3\begin{bmatrix} \overset{1\times 3}{\bar{a}_1^T} \\ \hline \underset{2\times 2}{A} \,\Big|\, \underset{2\times 1}{b} \end{bmatrix}} \qquad \begin{bmatrix} 1 \\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{bmatrix}$$

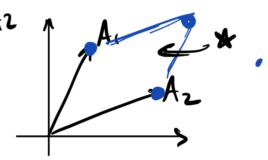
$$Ax = \begin{bmatrix} - \bar{a}_1^T - \\ \vdots \\ - \bar{a}_m^T - \end{bmatrix} x = \begin{bmatrix} A_1 \cdots A_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \bar{a}_1^T x \\ \vdots \\ \bar{a}_m^T x \end{bmatrix} = A_1 x_1 + \cdots + A_n x_n \} \rightarrow$$ linear comb of the cols of A

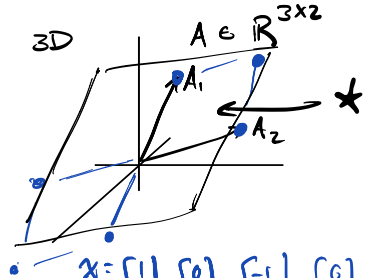
Examples: $A = [A_1 A_2]$

2D

$A \in \mathbb{R}^{2\times 2}$



3D $A \in \mathbb{R}^{3\times 2}$



Ax: $x = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$
$\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

$x = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}$

$\boxed{\text{Linear combs.}}$ $A \in \mathbb{R}^{m\times n}$

$\{ y \mid y = Ax,\ x \in \mathbb{R}^n \}$ (such that) $\rightarrow$ set of lin combs or a subspace $\} \rightarrow$ span of cols of A. "every possible lin comb"

for a particular y...

if $\exists\, x$ s.t. $y = Ax$ say "$y \in$ span of the cols of A"

Tech Defn: of Subspace vector space W, $V \subseteq W$

subspace is a set of vectors V (infinite / continuum)

s.t. if $A_1, A_2 \in V$ then $A_1 x_1 + A_2 x_2 \in V$ ← A_1, A_2 vectors, x_1, x_2 scalars

subspaces are sets of vectors that are "closed under linear combinations"

- all of W is a subspace
- 0 is in every subspace $A_1 0 + A_2 0 = 0$ ✓

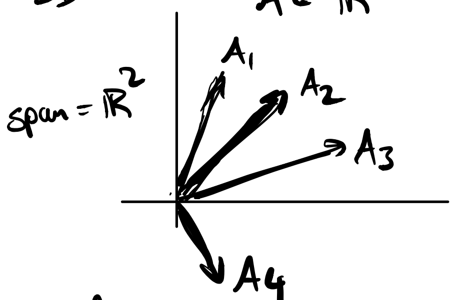
zero scalars

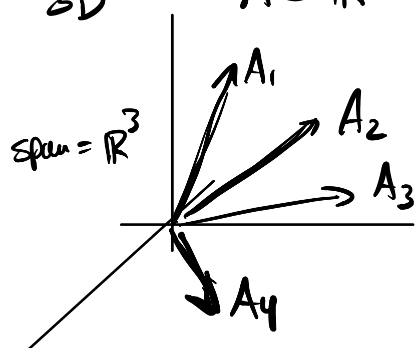
zero vector

$A = [A_1 \; A_2 \; A_3 \; A_4]$

3D $\quad A \in \mathbb{R}^{3\times 4}$

2D $\quad A \in \mathbb{R}^{2\times 4}$





More lin independent cols than dims

what is the span?

$\Rightarrow$ redundant vectors...

Tech Defn : Linearly Dep.

A_1 is lin dep on $\{A_2 \cdots A_n\}$

if $A_1 = A_2 y_2 + \cdots + A_n y_n$ for some $y_2 \cdots y_n$

set of vectors $[A_1 \cdots A_n]$ is lin dep

if $\exists \; x \neq 0 \Leftarrow$

$A_1 x_1 + \cdots + A_n x_n = 0$

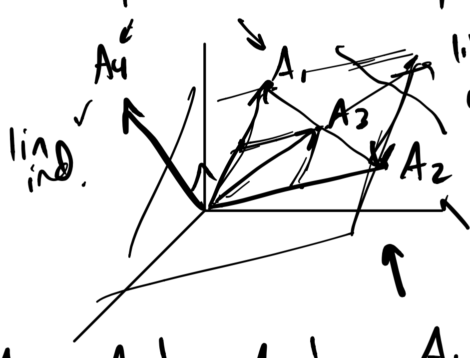
$$A_1 = A_2 \overbrace{\left(\frac{-x_2}{x_1}\right)}^{y_2} + \cdots + A_n \overbrace{\left(\frac{-x_n}{x_1}\right)}^{y_n}$$

$A = [A_1 \; A_2 \; A_3 \; A_4]$ cols are lin dep $\exists \; x \neq 0 \;\; \underline{Ax = 0}$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} \neq 0$

$$A_4 \frac{\cancel{x_4}}{x_4} = -A_1 \frac{x_1}{x_4} - A_2 \frac{x_2}{x_4} - A_3 \frac{x_3}{x_4}$$

if $x_i = 0$ for all x s.t. $Ax = 0$

$\Rightarrow A_i$ lin ind. of $A_j \; j \neq i$



lin dep. all on the same plane

$A_1 x_1 + A_2 x_2 + A_3 x_3 = 0$

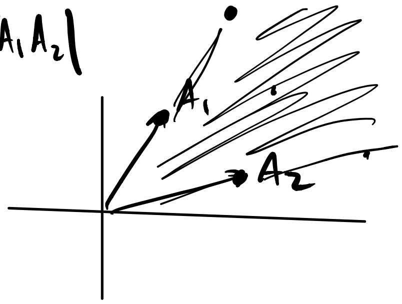
$[A_1, A_2 \; A_3]$ lin dep

$A_3 = A_1 \frac{1}{2} + A_2 \frac{1}{2} \qquad A_1 = 2A_3 - A_2 \qquad A_2 = 2A_3 - A_1$

"**Positive comb**" (cone)

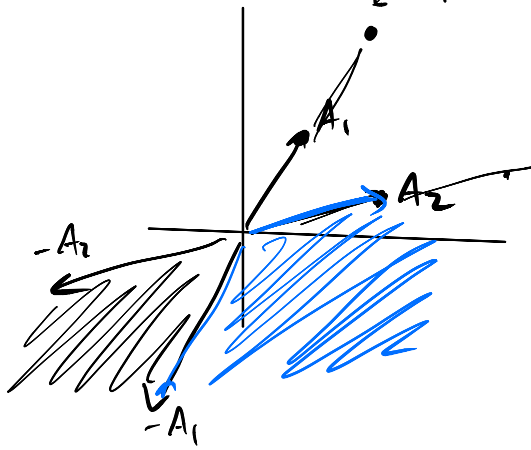
$\{y \mid y = Ax,\ x \geq 0,\ x \in \mathbb{R}^n\}$

$A = [A_1\ A_2]$



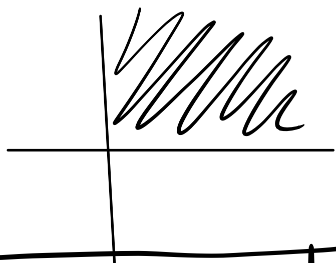
linear comb $\{y \mid y = Ax,\ x \in \mathbb{R}^n\}$

$A = [-A_1\ -A_2]$



$A = [-A_1\ A_2]$

if $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



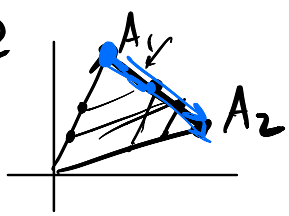
$\rightarrow \{y \mid y = Ix,\ x \geq 0\}$

"positive orthant"

$\boxed{\text{Convex comb}}$

$\{y \mid y = Ax,\ x \geq 0,\ \mathbb{1}^T x = 1\}$

2D



$A = [A_1\ A_2]$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$

$A_1 x_1 + A_2 x_2 \qquad \boxed{x_1 + x_2 = 1}$

$\downarrow$

$A_1 x_1 + A_1 x_2 - A_1 x_2 + A_2 x_2$

SIDE NOTES:

$\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$\mathbb{1}^T x = 1$

$[1 \cdots 1] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 1$

$\sum_i x_i = 1$

$\downarrow$

$\Delta_n = \{x \mid x \geq 0,\ \mathbb{1}^T x = 1,\ x \in \mathbb{R}^n\}$

"simplex in $\mathbb{R}^n$"

probability dist of dim n

$A_1(\overbrace{x_1+x_2}^{=1}) + (A_2-A_1)x_2$

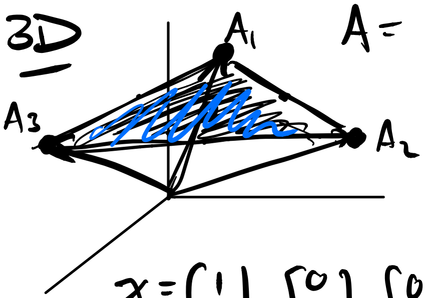
$A_1 + \underline{(A_2-A_1)}\,\underline{x_2} \qquad \underline{x_2} \in \underline{[0,1]}$

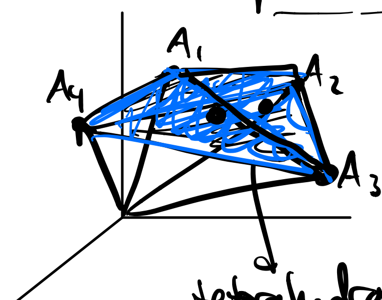
similarly

$A_2 + (A_1-A_2)x_1$

$A = [\underline{A_1 A_2} \; \underline{A_3 A_4}]$

3D $\qquad A = [A_1 \; A_2 \; A_3]$





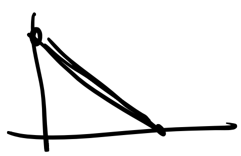
$$x = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}, \begin{bmatrix}0\\1/2\\1/2\end{bmatrix} \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$$

tetrahedron

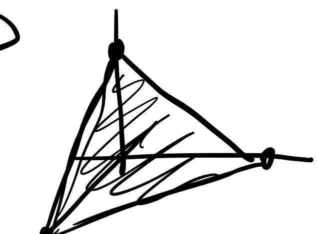
$$\begin{bmatrix}1/4\\1/4\\1/4\\1/4\end{bmatrix} \qquad \begin{bmatrix}1/3\\1/3\\1/3\\0\end{bmatrix}$$

simplex: $\quad A = I$

2D

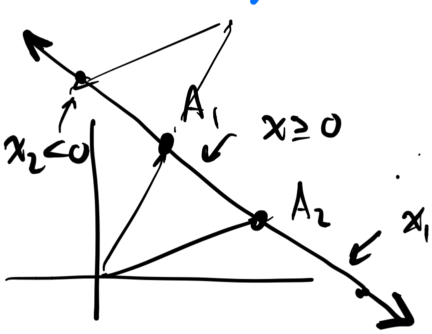


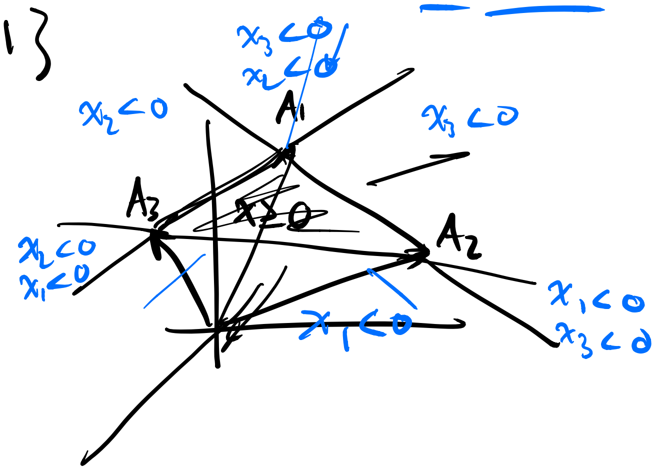
3D



$$\{\, y \mid y = Ax,\ \cancel{x \geq 0},\ \mathbb{1}^T x = 1 \,\}$$

NOT SURE:





$$x = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}$$

INNER PRODUCTS...

Adjoint Map:

$\langle y, x \rangle$

$\langle y, Ax \rangle = \langle A^* y, x \rangle$

Adjoint map of A $\Rightarrow$ transpose for real vectors

conjugate transpose for complex vectors

Ex. $x, y \in \mathbb{R}^n$

$y^T A x = (A^* y)^T x \Rightarrow A^* = A^T$

$\longrightarrow$ $\underline{f(\cdot), g(\cdot)}$ $A(f(\cdot))$

$\int \underline{g(t) A(f(t))} dt$ $\quad \underline{A^*}(g(\cdot))$

Transpose: $A_{ij} = (A^T)_{ji}$

- $(A^T)^T = A$
- $(ABC)^T = C^T B^T A^T$

$\Longrightarrow$ np.einsum

$\infty \left| \left[\quad \right]^{\infty} \right| \infty f(\cdot)$

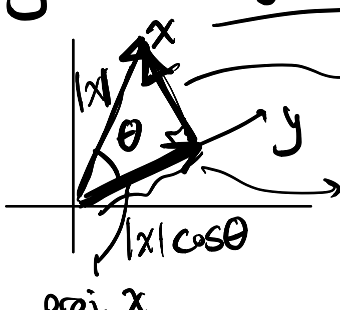
PROJECTIONS:

$|\cdot|$ – 2 norm

$$\text{proj}_y x = \frac{y}{|y|} |x| \cos\theta$$

$$y^T x = |x||y|\cos\theta$$

$$= \frac{y}{|y|^2} \underbrace{|y||x|\cos\theta}_{y^T x}$$



$\longrightarrow x - \text{proj}_y x$

$\longrightarrow |x| \cos\theta \frac{y}{|y|}$

$$\text{proj}_y x = \frac{1}{|y|^2} \underline{y y^T x} \Longleftarrow$$

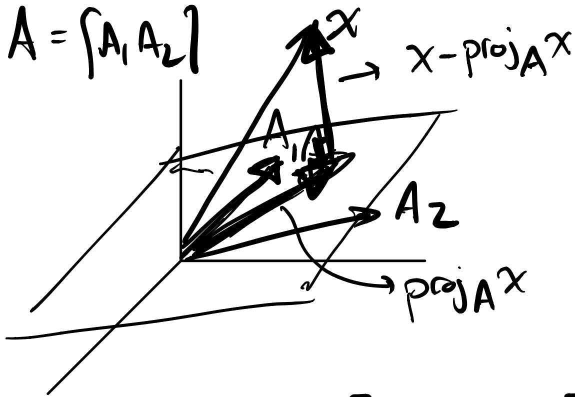
$$\left[\frac{1}{|y|^2} \begin{bmatrix} | \\ y \\ | \end{bmatrix} \begin{bmatrix} - y^T - \end{bmatrix} \right] x$$

outer product of y w/ itself

$$x - \text{proj}_y x = x - \frac{1}{|y|^2} \begin{bmatrix} | \\ y \\ | \end{bmatrix} \begin{bmatrix} - y^T - \end{bmatrix} x$$

$$= \left(I - \underline{\frac{1}{|y|^2} y y^T} \right) x$$

$A = [A_1\ A_2]$



$$\text{proj}_A\, x = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \underbrace{\left[\quad\right]}_{(A^TA)^{-1}} \begin{bmatrix} — A_1^T — \\ — A_2^T — \end{bmatrix} x$$

$$= \underset{\text{like } y}{A}\ \underset{\text{like } \frac{1}{|y|^2} = \frac{1}{((y^Ty)^{1/2})^2}}{(A^TA)^{-1}}\ A^T x$$

$$x - \text{proj}_A\, x = \left[I - A(A^TA)^{-1}A^T \right] x$$

Tech Defn:

projection

$\text{proj}_A(x)$

$$\boxed{\text{proj}_A(\text{proj}_A x) = \text{proj}_A x}$$

$$A(A^TA)^{-1}A^T\left(A(A^TA)^{-1}A^T x\right) = A(A^TA)^{-1}A^T x$$

$$\left(I - A(A^TA)^{-1}A^T\right)\left(I - A(A^TA)^{-1}A^T\right)x =$$

$$I - 2A(A^TA)^{-1}A^T + A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$$

$$A(A^TA)^{-1}A^T$$

$$I - A(A^TA)^{-1}A^T$$

SIDE NOTE: $A(A^TA)^{-1}A^T$ $\quad \cancel{A^T(A^TA)A}$

transposes usually alternate

$$A^T(AA^T)^{-1}A$$