Announcements

- COURSE EVALUATION
- COURSE ADVERTS
  - GAME THEORY COURSE EE546B
    LILLIAN RATLIFF Tu/Th 1-2:20PM
  - LINEAR SYS: EE 547 - SAM BURDEN
  - CONVEX OPTIMIZATION
  - ⇒ MARYAM'S CLASS EE578 — THEORETICAL RIGOROUS
    MWF 1130 → 12:50
  - DAN'S CLASS — 1. NETWORK FLOW
    - SHORTEST PATH + ROUTING GAMES
    Th 6-10PM
    2. MARKOV DECISION PROCESSES
    (STOCHASTIC EXTENSION)

Numerical Linear Algebra:

Matrix Norms:

measuring length: $\|\cdot\| : V \longrightarrow \mathbb{R}_+$

$x, y \in V$

- $|x+y| \leq |x| + |y|$ Triangle inequal.
- $|ax| = |a||x|$ $\longleftarrow$
- $|x| = 0 \Rightarrow x = 0$

Vector norms:

$|x|_p : \quad p \in [1, \infty]$

Matrix Norms: 2 different types

1. "as a vector"

applying vector norms to a matrix

$A \in \mathbb{R}^{m \times n}$

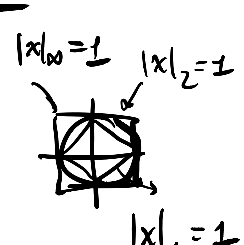
$|\mathrm{vec}(A)|_p$

" how big are the elements of A "

2. "as an operator"

induced norm

$A \in \mathbb{R}^{m \times n}$



$$|A|_{p,q} = \max_{|x|_p = 1} |Ax|_q$$

" how much does A increase the size of x."

$p, q \in [1, \infty]$

induced p-norm $\rightarrow$ $|A|_{p,p} = |A|_p$

2-norms for matrices

sweeping x around a unit ball (defined according to the $|\cdot|_p$ norm) in order to find the x that makes Ax the largest (in terms of the $|\cdot|_q$ norm

VECTOR 2 NORM

$|A|_F$ FROBENIUS NORM

treat A as vector..

$$|A|_F = \left(\sum_{ij} (A_{ij})^2\right)^{1/2}$$

$$= Tr(A^TA)^{1/2}$$

$$= Tr\left(V\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}U^TU\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T\right)^{1/2}$$

$$= Tr\left(V\begin{bmatrix}\Sigma^2 & 0\\ 0 & 0\end{bmatrix}V^T\right)^{1/2}$$

$$= Tr\left(\begin{bmatrix}\Sigma^2 & 0\\ 0 & 0\end{bmatrix}\underbrace{V^TV}_{I}\right)^{1/2}$$

$$= Tr\begin{bmatrix}\Sigma^2 & 0\\ 0 & 0\end{bmatrix}^{1/2}$$

$$= \left(\sum_{i=1}^{k} \sigma_i^2\right)^{1/2}$$

$$|A|_F = |\bar{\sigma}|_2 \leftarrow \checkmark$$

$$\bar{\sigma} = [\sigma_1 \cdots \sigma_k]$$

$$|A|_F \geq |A|_{2,2}$$

INDUCED 2-NORM

$$|A|_{2,2} = \max_{|x|_2=1} |Ax|_2$$

for $A \in \mathbb{R}^{m\times n}$ $\quad A = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T$

$$|A|_{2,2} = \max_{|x|_2=1} (x^TA^TAx)^{1/2}$$

$$\sigma_{max} = \max_{|x|_2=1} \left(x^TV\begin{bmatrix}\Sigma^2 & 0\\ 0 & 0\end{bmatrix}V^Tx\right)^{1/2}$$

$$x^T\begin{bmatrix}v_1 \cdots v_n\end{bmatrix}\begin{bmatrix}\begin{matrix}\sigma_1^2 & & 0\\ & \ddots & \\ 0 & & \sigma_k^2\end{matrix} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1^T\\ \vdots\\ v_n^T\end{bmatrix}x$$

$$\left(\begin{bmatrix}1 \cdots 0\end{bmatrix}\begin{bmatrix}\sigma_1^2 & \\ & \ddots\end{bmatrix}\begin{bmatrix}1\\ 0\\ \vdots\end{bmatrix}\right)^{1/2}$$

$$|A|_{2,2} = \underline{\sigma_{max}} \leftarrow$$

max singular value

Refer to animation of SVD on Wikipedia

Numerical Linear Algebra

Pioneers. / computing
1950's on
- von Neumann
- Turing
- Householder
- Knuth

LAPACK / BLAS]
- FORTRAN

Computing DECOMPOSITIONS

- Inverse of Matrix $y = Ax$ ⟸
- Singular Value Decomposition
- QR decomposition ⟸
- LU decomposition ⟸
- Eigenvalue decomposition

Goal: find a way to compute these things easily by hand / tell a computer what to do

Matrix Inverse $A \in \mathbb{R}^{n \times n}$

$$y = Ax \Rightarrow x = A^{-1} y$$

Elementary Matrices / Gaussian Elimination

$$E_k \cdots E_1 A = I$$

k row reduction operations

E_i for replacing any row w a linear combination of rows

"pivots

$$(E_k \cdots E_1) = A^{-1}$$

$$A(E_k \cdots E_1) = I$$

k col reduction operations

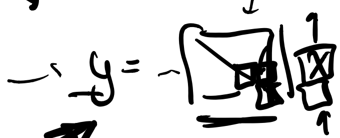
$$(E_k \cdots E_1) = A^{-1}$$

LU decomposition

L: "lower triangular"

partial Gaussian

U: "upper triangular"



$$L_k \cdots L_1 A = U \Rightarrow \boxed{A = LU}$$

elementary matrices, lower triangular

lower triangular constructed from elementary matrices

upper triangular

$$L^{-1} = L_k \cdots L_1$$

← lower triangular

L ← lower triangular

$$A^{-1} = U^{-1} L^{-1}$$

inverses of triangular matrices are easy to compute

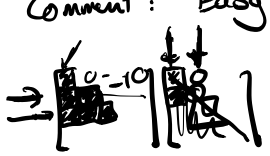
Lemma L_1, L_2 ↔ lower triangular

$L_1 L_2 =$  $= \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & 0 & \cdots \end{bmatrix}$

Lemma A can be row reduced to U, by lower triangular elementary matrices $L_1, \ldots, L_k$

$$L_1 A = \begin{bmatrix} U \end{bmatrix}$$

Comment: Easy to invert L

 $= \begin{bmatrix} 1 & & 0 \\ 0 & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$ $\quad L^{-1} = \begin{bmatrix} \square & 0 & \cdots & 0 \\ & & & \end{bmatrix}$

QR DECOMPOSITION:

SIDE NOTE
R IS NOT THE ROTATION MATRIX

$$A = \underset{\text{orthonormal matrix}}{Q} \; \underset{\text{upper triangular}}{R}$$

2 METHODS TO COMPUTE
- GRAM SCHMIDT → "orthonormalize a basis"
- HOUSEHOLDER REFLECTIONS

QR DECOMPOSITION IS FINDING AN ORTHONORMAL BASIS FOR COLSPACE OR RANGE OF A.

$$Q = [Q_1 \cdots Q_m] \quad R = [R_1 \cdots R_n]$$

$$A \in \mathbb{R}^{m \times n} \quad A = [A_1 \cdots A_n]$$

$$[A_1 \cdots A_n] = [Q_1 \cdots Q_m][R_1 \cdots R_n]$$

range

orthonormal

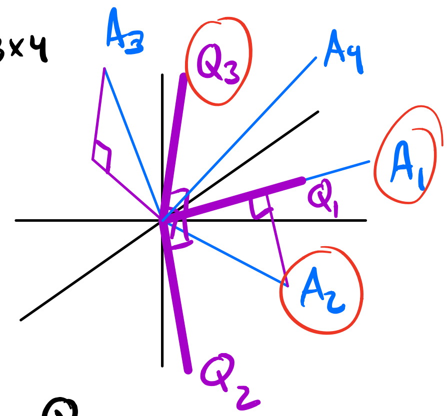
coeffs or coords of A_1 wrt to cols of Q

coords of A_n wrt cols of Q

take cols of A → do Gram Schmidt → cols of Q

1. GRAM SCHMIDT

$$A = [A_1 \cdots A_n] \qquad A = \mathbb{R}^{3\times 4}$$



$$Q_1 = A_1/|A_1|$$

$$Q_2 = A_2 - Q_1 Q_1^T A_2 \qquad Q_2 \Leftarrow Q_2/|Q_2|$$

$$(I - Q_1 Q_1^T) A_2$$

$$Q_3 = \left(I - \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}\right) A_3 \qquad Q_3 \Leftarrow \frac{Q_3}{|Q_3|}$$

could do

$$\left(I - \begin{bmatrix} A_1 & A_2 \end{bmatrix} \left(\begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix}\right)^{-1} \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix}\right) A_3$$

and so on..

$$Q_4 = \left(I - \begin{bmatrix} Q_1 & Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix}\right) A_4 = 0 \qquad Q_4 \Leftarrow \frac{Q_4}{|Q_4|}$$

QR Decomp

upper triangular $\quad Q^T A = R$

$$IA =$$

$$\underbrace{Q Q^T}_{I} A = Q\, Q^T A$$

$$\begin{bmatrix} Q_1^T \\ \vdots \\ Q_n^T \end{bmatrix} \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$$

$$\begin{bmatrix} Q_1^T A_1 & \cdots & & Q_1^T A_n \\ \cancel{Q_2^T A_1} \; 0 & Q_2^T A_2 & \cdots & Q_2^T A_n \\ \cancel{Q_3^T A_1} \; 0 & \cancel{Q_3^T A_2} \; 0 & Q_3^T A_3 & \cdots \end{bmatrix}$$

$$A = QR$$

where $R = Q^T A \leftarrow$ upper triangular

2 Householder Reflections

$$\underbrace{H_k \cdots H_1}_{\text{Householder reflections}} A = R \rightarrow \text{upper triangular}$$

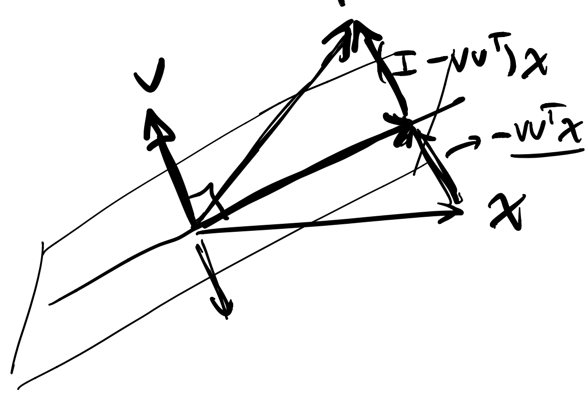
$$\underbrace{H_k \cdots H_1}_{\text{orthonormal matrix}} \qquad Q = (H_k \cdots H_1)^{-1} = (H_k \cdots H_1)^T$$

$$A = QR = (H_k \cdots H_1)^T R$$

H_j : Householder reflection

unit vector v

$$H_j = I - 2vv^T \rightarrow \text{related to projections}$$



PROJECTION:

$(I - vv^T)x$

Householder reflection

$(I - 2vv^T)x$

Properties of H_j

- Symmetric : $H_j^T = H_j$
- unitary : $H_j^T H_j = I$ rotations / reflections

$$(I - 2vv^T)^T (I - 2vv^T) = I$$

$$I - 4vv^T + 4v\underbrace{v^T v}_{1}v^T = I$$

- involutory: $H_j^2 = I$
- eigenvalues : ± 1 all eigenvalues except one are 1 last is -1
- determinant : $\det(H_j) = -1$

Computing QR decomp

$$H_1 = I - 2vv^T \quad v = \frac{A_1 - |A_1| e_1}{\left[(A_1 - |A_1| e_1)^T (A_1 - |A_1| e_1)\right]^{1/2}}$$

$$H_1 A = H_1 \left[A_1 \cdots A_n\right]$$

$$= \left[H_1 A_1 \ \cdots \ H_1 A_n\right]$$

$$H_1 A_1 = \left[I - 2\frac{(A_1 - |A_1| e_1)(A_1 - |A_1| e_1)^T}{\left[(A_1 - |A_1| e_1)^T (A_1 - |A_1| e_1)\right]}\right] A_1$$

$$= A_1 - 2\frac{(A_1 - |A_1| e_1)(A_1^T A_1 - |A_1| e_1^T A_1)}{A_1^T A_1 - 2|A_1| e_1^T A_1 + |A_1|^2}$$

$$= A_1 - \frac{\cancel{2}(A_1 - |A_1|e_1)\cancel{|A_1|(|A_1| - e_1^T A_1)}}{\cancel{2|A_1|(|A_1| - e_1^T A_1)}}$$

$$= A_1 \cancel{- A_1} + |A_1|e_1 = |A_1|e_1$$

$$H_1 A = \begin{bmatrix} |A_1| & \cdots \\ 0 & \\ \vdots & u \\ 0 & \end{bmatrix} \qquad v = \frac{u}{|u|}$$

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & I - 2vv^T \end{bmatrix}$$

$$H_2 H_1 A = \begin{bmatrix} |A_1| & * & \sim \\ 0 & |u| & \sim \\ \vdots & 0 & \\ 0 & 0 & \boxed{\times} \end{bmatrix}$$

and so on

$$\underbrace{H_k \cdots H_1}_{\text{unitary}} A = R.$$

$$A = QR \quad \text{where } Q = (H_k \cdots H_1)^T$$

- Solving Equations $\rightarrow$ computing an inverse.
  - $\rightarrow$ triangular, orthonormal $\}$ $\rightarrow$ easy to invert.

$A = LU \qquad A^{-1} = U^{-1}L^{-1}$

$A = QR \qquad A^{-1} = R^{-1}Q^{T}$

- function (analytic) of A. $\leftarrow$ $\sigma(f(A)) = \{f(\lambda_1) \dots f(\lambda_n)\}$

  eigenvalue decomp of $A \qquad A = PJP^{-1}$

- scaling of a matrix

  SVD $\qquad A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$

  $\hookrightarrow$ DECOMPOSING DOMAIN & CODOMAIN INTO $R(A)$ $R(A^T)$ $N(A^T)$ $N(A)$

$$A = \begin{bmatrix} \underset{\text{basis for } R(A)}{U_1} & \underset{\text{basis for } N(A^*)}{U_2} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} \begin{matrix} \rightarrow \text{basis for } R(A^*) \\ \rightarrow \text{basis for } N(A) \end{matrix}$$

$= U_1 \Sigma V_1^*$

$\underbrace{[U_1]}_{\text{orthonormal}} [\Sigma V_1^*]$

$\Sigma V_1^* = Q'R$

$A = QR$

$= \underbrace{U_1 Q'}_{Q} R$

$\chi_A(s) \qquad \chi_A(A) = 0 \iff \chi_A(\lambda) = 0$

$A^n = -\alpha_{n-1}A^{n-1} - + \dots - \alpha_1 A - \alpha_0 I$

Controllability : range of $e^{At}B$

547 $\qquad \rightarrow \dot{x} = Ax + Bu$

Cayley Hamilton

$R(e^{At}B) = R(\underbrace{[A^{n-1}B \dots AB \; B]}_{\text{controllability matrix}})$

Gregory Perelmann

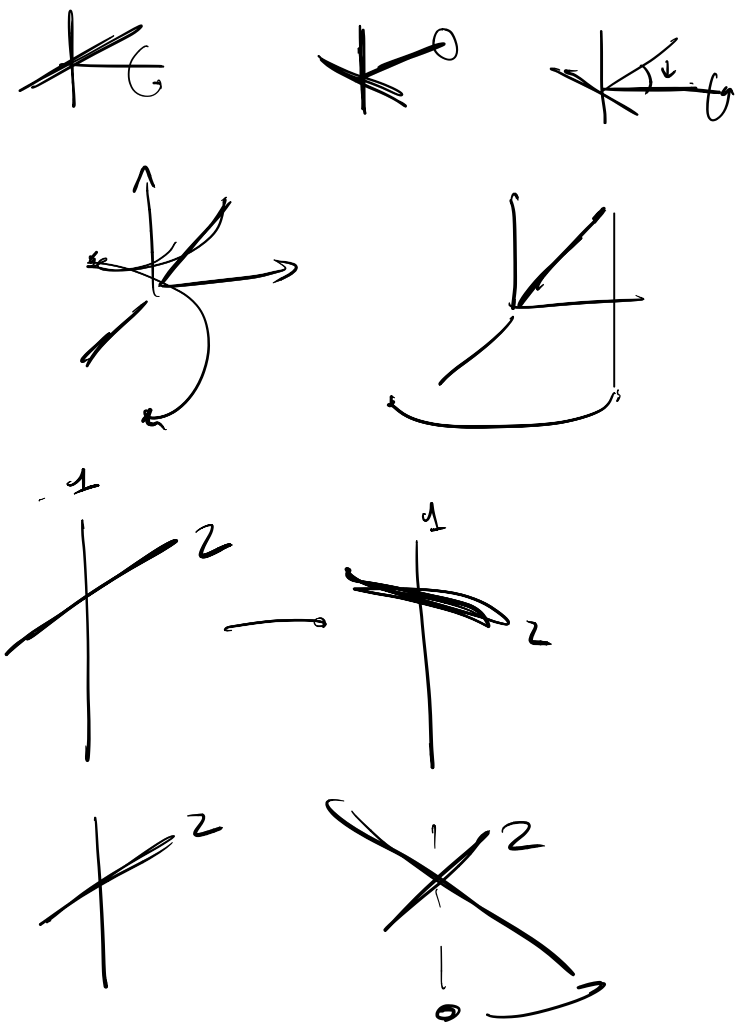
$[L_1 | A =$

$L_1 [\boxed{A_1} \cdots A_n] = [\quad]$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

L P U

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}$$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