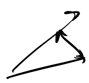
Review:

magnitude of a function

$f : [0,1] \longrightarrow \mathbb{R}$ $\quad |f|_2 = \left(\int_0^1 f(t)^2 \, dt \right)^{1/2}$

$|f|_1 = \int_0^1 |f(t)| \, dt$

$|f|_{10} = \left(\int_0^1 f(t)^{10} \, dt \right)^{1/10}$

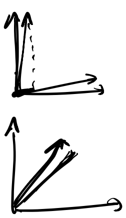


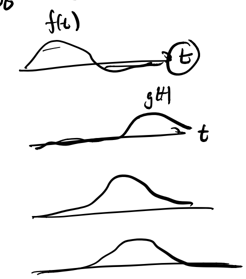
$x, y \quad y^T x = \sum_i x_i y_i \qquad \langle f, g \rangle = \int_0^1 f(t) g(t) \, dt \leftarrow$ small

$f : [0,1] \Rightarrow \mathbb{R}$

$g : [0,1] \Rightarrow \mathbb{R}$

x_{i}





Vector Derivatives:

what is a derivative?

$x \, , \, f(x)$ $\quad \boxed{\underbrace{\Delta f}_{\substack{\text{perturb} \\ \text{in func.}}} = \underbrace{\frac{\partial f}{\partial x}}_{\substack{\text{linear} \\ \text{map.} \to \text{matrix}}} \underbrace{\Delta x}_{\substack{\text{perturb} \\ \text{variable}}}}$

$\frac{\partial f}{\partial x}$ Leibnitz

$\partial f = \frac{\partial f}{\cancel{\partial x}} \cancel{\partial x}$

- $f : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad \Delta f = \frac{\partial f}{\partial x} \Delta x$
- $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad \underbrace{\Delta f}_{\text{scalar}} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}}_{\frac{\partial f}{\partial x} \text{ row vector}} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} = \frac{\partial f}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f}{\partial x_n} \Delta x_n$

if x is a column vector & $f(x)$ is scalar

$\frac{\partial f}{\partial x} \Rightarrow$ row vector

$\frac{\partial f}{\partial x} \Rightarrow \nabla f \to$ col vector

- $f : \mathbb{R} \longrightarrow \mathbb{R}^m$ $\quad \begin{bmatrix} \Delta f_1 \\ \vdots \\ \Delta f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \vdots \\ \frac{\partial f_m}{\partial x} \end{bmatrix} \Delta x$

$\frac{\partial f}{\partial x}$

- $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad \begin{bmatrix} \Delta f_1 \\ \vdots \\ \Delta f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$

Product rule & chain rule

- $f(x) = c^T x \implies \Delta f = c^T \Delta x \quad \frac{\partial f}{\partial x} = c^T \quad c \in \mathbb{R}^n$
- $f(x) = Ax \longrightarrow \Delta f = A \Delta x \quad \frac{\partial f}{\partial x} = A \quad A \in \mathbb{R}^{m \times n}$
- $f(x) = x^T Q x \quad \Delta f = \underbrace{\Delta x^T Q x}_{\text{transpose}} + x^T Q \Delta x$ } → Product rule

$$= x^T Q^T \Delta x + x^T Q \Delta x \qquad \frac{\partial}{\partial x} f(x) g(x) = \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial x} = x^T (Q^T + Q) \Leftarrow = x^T (Q^T + Q) \Delta x$$

Same if Q is symmetric...

" $\frac{\partial f}{\partial x} = 2x^T Q$ "

What if Q is not sym: $x^T Q x = x^T \left(\frac{1}{2}(Q + Q^T) + \frac{1}{2}(Q - Q^T) \right) x$

if Q has a piece that is skew sym. it desn't affect $x^T Q x$ $\leftarrow$ $Q = \underbrace{\frac{1}{2}(Q + Q^T)}_{\text{sym.}} + \underbrace{\frac{1}{2}(Q - Q^T)}_{\text{skew sym}}$

"averaging Q & its transpose"

$(Q - Q^T)^T = -Q + Q^T$

$$x^T \frac{1}{2}(Q - Q^T) x$$

sym/skew sym decomposition

$$\frac{1}{2} x^T Q x - \frac{1}{2} \underbrace{x^T Q^T x}_{x^T Q x} = 0$$

K: skew sym $K = -K^T$

$$\Rightarrow x^T K x = 0$$

• $f(x) = \sin(x^TQx) + \ln(c^Tx)$

$u = x^TQx$

$u' = c^Tx$

$\Delta f = \frac{\partial \sin}{\partial u}\frac{\partial u}{\partial x}\Delta x + \frac{\partial \ln}{\partial u'}\frac{\partial u'}{\partial x}\Delta x$

$= \underbrace{\cos(x^TQx)}_{\text{scalar}}\, x^T(Q+Q^T)\Delta x + \frac{1}{c^Tx}c^T\Delta x$

$= \Big[\underbrace{\cos(x^TQx)}_{\text{scalar}}\, x^T(Q+Q^T) + \frac{1}{c^Tx}c^T\Big] \to$ row vector

row vec $\quad \frac{\partial f}{\partial x} \quad$ mat. $\quad$ scalar $\quad$ row

• $\frac{\partial^2 f}{\partial x^2}$: $f(x) = x^TQx \Rightarrow \Delta\frac{\partial f}{\partial x} = \Delta x^T(Q+Q^T)$

$\Delta\left(\frac{\partial f}{\partial x}\right) = \Delta\left(x^T(Q+Q^T)\right) = \Delta x^T\boxed{(Q+Q^T)} \to \frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial^2 f}{\partial x^2} = \underset{\text{2nd perturb}}{} \overset{\text{1st perturb}}{\left[\,Q + Q^T\,\right]} \qquad \Delta f \qquad \frac{\partial f}{\partial x}\Delta x$$

$\Delta\left(\frac{\partial f}{\partial x}\right) = \Delta x^T\frac{\partial^2 f}{\partial x^2}$

Taylor exp:

$$f(x+\Delta x) \approx f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2}\Delta x^T\underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{matrix}}\Delta x + \cdots$$

$f(x) + \Delta f$

$\frac{\partial f}{\partial x}\Delta x$

$\left(\frac{\partial f}{\partial x} + \Delta\frac{\partial f}{\partial x}\right)\Delta x = \frac{\partial f}{\partial x}\Delta x + \Delta x^T\frac{\partial^2 f}{\partial x^2}\Delta x$

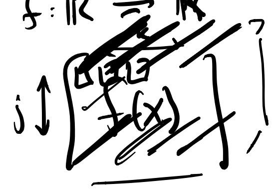
$f: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad \frac{\partial f}{\partial x} = m\left[\ {}^{n}\ \right]$ $\qquad \Delta f = \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}}_{\text{matrix}} \Delta x_i$

$f: \mathbb{R}^n \to \mathbb{R}^{m \times p}$

$\frac{\partial f}{\partial x} = m\left[\ {}^{p}\ \right]\Delta x_1 + \dots + {}^{p}\left[\ \right]\Delta x_n$

$= m$  Δx $\leftarrow$

$f: \mathbb{R}^n \to \mathbb{R}^{m \times p}$



$f(x)_{jk} \qquad \frac{\partial f(x)_{jk}}{\partial x} = \left[\frac{\partial f_{jk}}{\partial x_1} \cdots \frac{\partial f_{jk}}{\partial x_n}\right] \ \forall jk$

General Relativity

Diff. geometry

Einstein summation notation

$\frac{\partial f_{11}}{\partial x} = [\ \text{———}\ ]$

$\frac{\partial f_{12}}{\partial x} = [\ \text{———}\ ]$

$\frac{\partial f_{13}}{\partial x} = [\ \text{———}\ ]$

$M_{ijk} x_i \Rightarrow$ "sum over i"

numpy function: $\boxed{\text{numpy.einsum}}$ $\leftarrow$ super useful for array manipulation

import numpy as np.

- np.einsum('ij,jk', A,B) ⇒ matrix multiplication
- np.einsum('ij,j', A,x) ⇒ right mult. by a vector Ax
- np.einsum('ij,i', A,x) ⇒ left mult x^TA
- np.einsum('ij,jk,kl', A,B,C) etc.

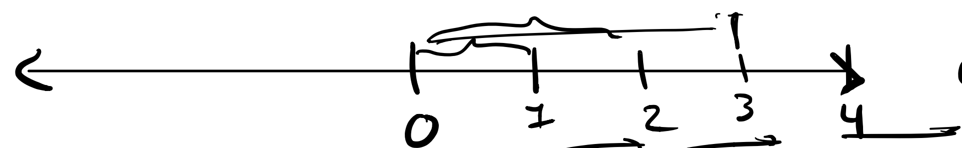
Complex Analysis Review: even matrices $\bar{w}$ all real entries can have complex eigenvalues

Rotation matrices

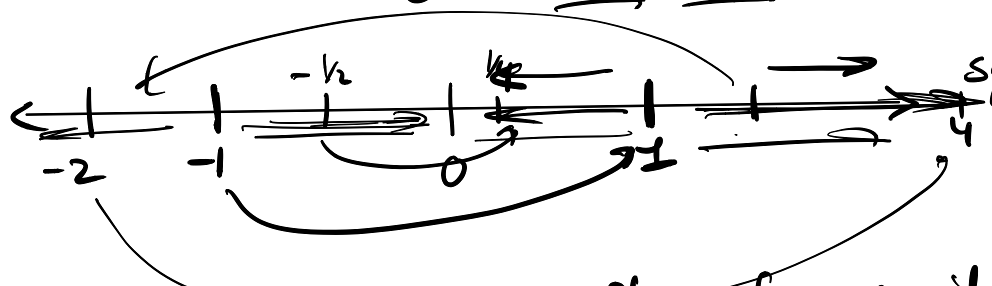
Rotation / oscillation at heart of complex #'s

polynomials $\bar{w}$ real coeffs can have complex roots

Real # line



addition



squaring

multiplying by negative flips from 1 side to the other

2^2 = start 1, multiply by 2, multiply by 2 $\Rightarrow$ 4

3^2 = start 1, mult. by 3, mult by 3 $\Rightarrow$ 9

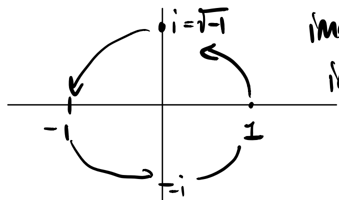
$\sqrt{4}$? mult by $z=2$, mult. by $z=2$ $\Rightarrow$ 4

$\sqrt{9}$? mult. by $z=3$, mult by $z=3$ $\Rightarrow$ 9

$\sqrt{-1}$? mult. by z, mult. by z $\Rightarrow -1$

what is z? z can't be pos $\Rightarrow$ stay pos

z can't be neg $\Rightarrow$ become pos want z be half a flip from 1 to -1



imaginary #'s inherently have this idea of rotation

complex plane

can't rotate on a # line

Complex # Representation:

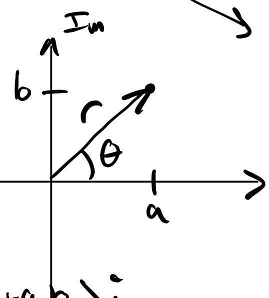
- $z = a+bi \rightarrow$ addition
- $z = re^{i\theta}$ $\Big\}$ multiplication
  $r > 0$

$(a_1+b_1 i)+(a_2+b_2 i)$
$= (a_1+a_2)+(b_1+b_2)i$

Carte.
$(a_1+b_1 i)(a_2+b_2 i)$

$a_1a_2 + (a_2b_1+a_1b_2)i$
$+ b_1b_2 i^2$

$= a_1a_2 - b_1b_2 + (a_2b_1+a_1b_2)i$



Converting between forms

$z = re^{i\theta} \rightarrow r\cos\theta + r\sin\theta\, i$

$z = a+bi \rightarrow \sqrt{a^2+b^2}\, e^{i\,\operatorname{atan}\frac{b}{a}}$

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = \underbrace{r_1r_2}_{\text{mag. mult.}}e^{i\overbrace{(\theta_1+\theta_2)}^{\text{phases add.}}}$$

$\sqrt{a^2+b^2} = |z|$
$= \sqrt{z^* z}$

compare

$*$ conjugate
$(a+bi)^* = a-bi$
$z^*z = (a+bi)^*(a+bi)$
$= (a-bi)(a+bi)$
$= a^2+b^2$

$\|x\|_2 = \sqrt{x^Tx}$

$x \in \mathbb{C}^n$

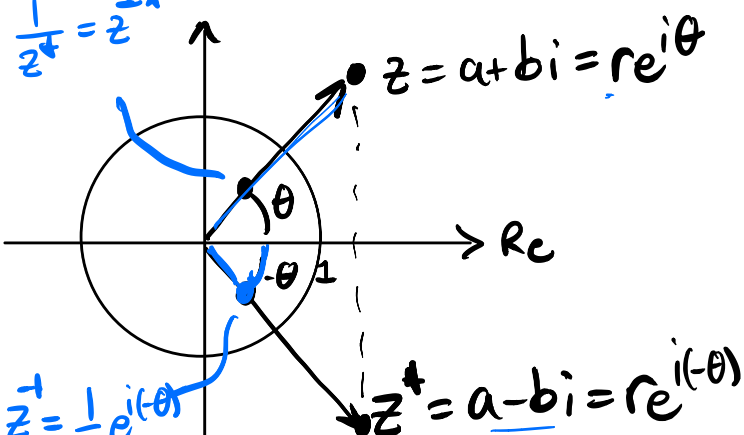
$\|x\|_2 = \sqrt{x^*x}$

for vectors

$*$: conjugate transpose

(transpose & conjugate complex #'s)

$\frac{1}{z^*} = z^{-*}$



$z^{-1} = \frac{1}{r}e^{i(-\theta)}$

$= \frac{1}{a+bi}\frac{(a-bi)}{(a-bi)} = \frac{a-bi}{a^2+b^2}$

when you invert a complex #
1: invert the magnitude ("flip" across unit circle)
2: invert the angle (negate the angle)

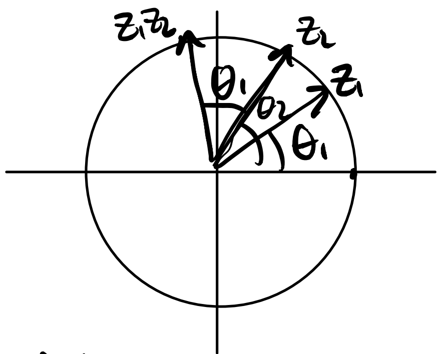
Euler's Formula:

$re^{i\theta} = r\cos\theta + r\sin\theta\, i$

Most famous

$\underbrace{e^{i\pi}}_{-1} + 1 = 0$

Roots of Unity:



multiplying 2 complex #'s on the unit circle... stay on the unit circle

$z = re^{i\theta} \Longleftrightarrow r = 1$

$z_1 = e^{i\theta_1} \quad z_2 = e^{i\theta_2}$

$z_1 z_2 = e^{i(\theta_1 + \theta_2)}$

Solutions to:

$z^n = 1$

- raise to $n \to$ doing several multiplications

for n: algebraically

solution $\quad 1 = e^{i2\pi} \leftarrow$ 1 rotation $\quad z^n = e^{i2\pi} \Rightarrow z = e^{\frac{i2\pi}{n}}$

$1 = e^{(i2\pi)2} \leftarrow$ 2 rotations $\quad z^n = e^{i2\pi 2} \Rightarrow z = e^{\frac{i2\pi 2}{n}}$

heart DFT

solutions $\quad 1 = e^{ik2\pi} \leftarrow$ k rotations $\rightarrow \quad z = e^{\frac{ik2\pi}{n}}$

$\boxed{e^{\frac{ik2\pi}{n}} \text{ for } k = 0, 1, 2, \ldots, n-1} \Rightarrow e^{ik2\pi/n}$ for $k = \ldots -2, -1, 0, 1, 2 \ldots$

Now. $k = n \Rightarrow e^{\frac{in2\pi}{n}} = e^{i2\pi} = 1$ same as $k = 0$

$k = n+1 \Rightarrow e^{\frac{i(n+1)2\pi}{n}} = \cancel{e^{\frac{in2\pi}{n}}} e^{i2\pi/n}$ same as if $k = 1$

$k = n+2 \Rightarrow$ same as $k = 2$

what if k is negative?

$k=-1$ $\quad e^{i(-1)\frac{2\pi}{n}}1 = e^{i(-1)\frac{2\pi}{n}} e^{in\frac{2\pi}{n}} = e^{i(n-1)\frac{2\pi}{n}}$

$\rightarrow k = n-1$

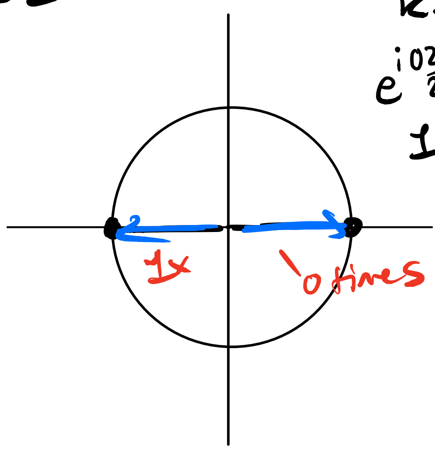
$k=-2$ $\quad e^{i(-2)2\pi/n} = e^{i(-2)\frac{2\pi}{n}} e^{in2\pi/n} = e^{i(n-2)2\pi/n}$

$\rightarrow k = n-2$

$n=2$ $\quad k=0,1$

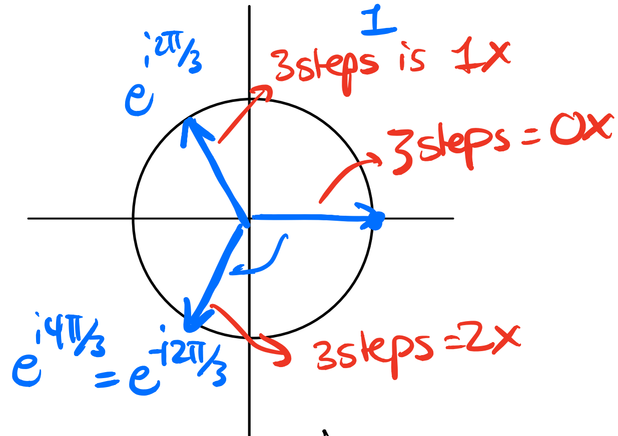
$e^{i0\frac{2\pi}{2}}, e^{i\frac{2\pi}{2}}$

$1, -1$

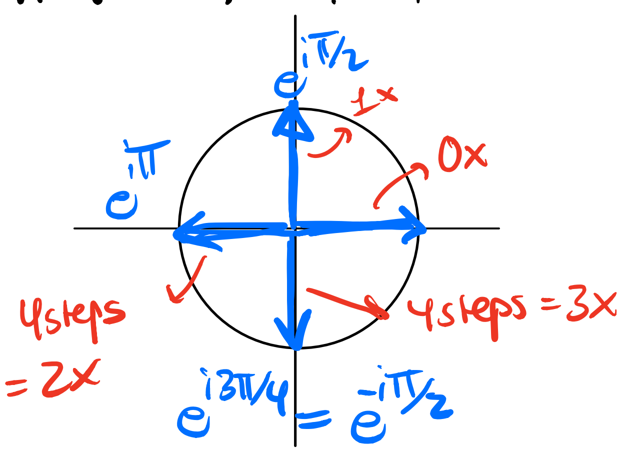


$n=3$ $\quad e^{i0\frac{2\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i2(2\pi)/3}$

1



$n=4$ $\quad 1, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/4}$



$n=5$ $\quad 1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}$