TOPICS:
- PROOFS
- BASES

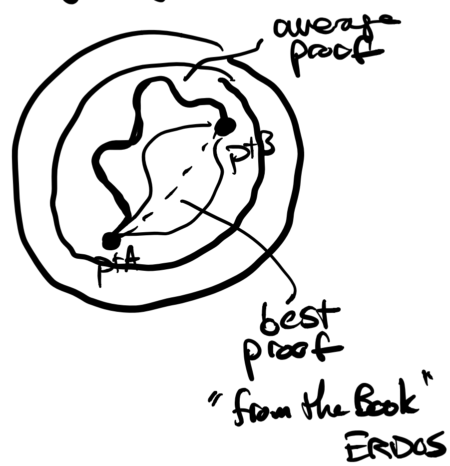
PROOFS:

"know how everything connects to everything" :)

Following a proof: traveling from A to B

FINDING a proof: seeing a big section of the globe.

Classes are rough and so is homework...



TECHNIQUES:

1. know a lot of tricks / pattern match → get easier w/ practice
   (memorize it)
   move things around algebraically.
2. Draw a picture: ★ I pretty much always do this.
   (geometric intuition)
   (spatial intuition)
   Ex. vector fields → fluids
   PDES → electromagnetic
3. Physical / Real world example → physicists
4. Look for counter examples
   "try to break the statement" → indirect proof
   if you can't construct a counter example
   ⇒ that intuition often leads to a proof.

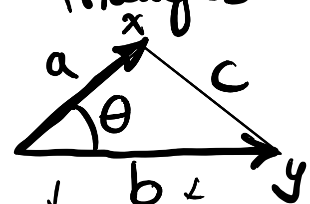
Ex H1Q1A prove $x^Ty = |x||y|\cos\theta$

what are we given defn: x^Ty, law of cosines

$x^Ty = \sum_i x_i y_i$; law of cosines → trigonometry triangles

- connect triangle w/ vectors (match up θ)

(1)



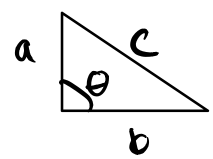
- write a, b, c in terms of x, y

$|a| = |x|$, $b = |y|$

$|c| = |x - y|$

→ $|c|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$

Note: extension of Pythag. Thm.



$\cos\theta = 0$

$|c|^2 = |a|^2 + |b|^2$

- norms → inner products

→ $|a|^2 = x^Tx$, $|b|^2 = y^Ty$

$|c|^2 = (x-y)^T(x-y)$

$= x^Tx + y^Ty - 2y^Tx$

$|c|^2 = |a|^2 + |b|^2 - 2\underline{y^Tx}$

$y^Tx = |x||y|\cos\theta$

Note: coming up w/ solution → meandering

writing it out... should flow from start to finish

WTS: $x^Ty = |x||y|\cos\theta$

→ $(x-y)^T(x-y) = x^Tx + y^Ty - 2y^Tx$

by the law of cosines

→ $(x-y)^T(x-y) = x^Tx + y^Ty - 2|x||y|\cos\theta$

⇒ $2y^Tx = 2|x||y|\cos\theta$

Diagrams:

- google drawings
- powerpt/keynote
- ipad
- photo
- tikz (latex)

Symbols:

"implies"	"implies both ways"	"not"	"and"	"or"	"contradiction"
$\Rightarrow$	$\Longleftrightarrow$	$\neg, \sim, !$	$\wedge, \&$	$\vee, \|\|$	$\perp$

Quantifiers:

$\forall$ "for all" "for ea." "for any" — every element in a set

$\exists$ "there exists" — at least one element in a set

$\exists!$ "there exists uniquely" — only 1 element in a set

LOGICAL STATEMENTS:

Statement ← SAME → **Contrapositive**

if p, then q

$\Rightarrow$ $p \Rightarrow q$

"p is sufficient for q"

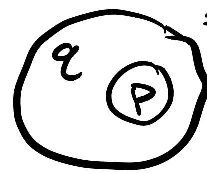
"inside p $\Rightarrow$ inside q"

NOT SAME $\updownarrow$

if $\neg q$, then $\neg p$

$\neg q \Rightarrow \neg p$

(sufficient) "outside q $\Rightarrow$ outside p"

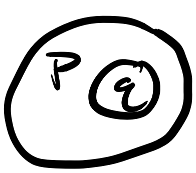


Inverse ← SAME → **Converse** (contrapositive of inverse)

if $\neg p$, then $\neg q$

$\neg p \Rightarrow \neg q$

"p is necessary for q



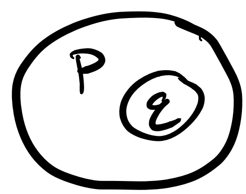
"outside p $\Rightarrow$ outside q"

if q, then p

$q \Rightarrow p$

(necessary)

"inside q $\Rightarrow$ inside p"



Statements that go both ways

p if and only if q $\quad$ q if and only if p $\quad$ $p \Leftrightarrow q$

converse $\quad$ contrapositive $\quad$ statement $\quad$ inverse

p iff q $\quad$ q iff p $\quad$ p, q

p, q are logically equivalent

To prove: prove both directions

1. $\Rightarrow$) (sufficient) $\quad$ prove statement or contrapositive

and

2. $\Leftarrow$) necessary $\quad$ prove inverse or converse

iff also written as "necessary and sufficient"

More Logic – Truth Tables

	p	$\neg p$
q		Not allowed
$\neg q$	Not allowed	

$p \Rightarrow q$

$q \Rightarrow p$

AND statements

	p	$\neg p$
q	$p \wedge q$	$q \wedge \neg p$
$\neg q$	$\neg q \wedge p$	$\neg q \wedge \neg p$

OR STATEMENTS

	p	$\neg p$
q		
$\neg q$		

$p \vee q$

$\neg p \vee \neg q$

Two Other Important Techniques

- proof by contradiction
- proof by induction.

Proof by contradiction:

Examples to come.

direct proof: $p \Rightarrow q$

contrapositive proof: $\neg q \Rightarrow \neg p$

proof by contradiction $p, \neg q \Rightarrow \perp$

Proof by induction: Examples to come

want to show q_k something for all natural #s $k = 0, 1, 2, \ldots$

① prove q_0 (base case)

② if q_k, then q_{k+1} $\quad q_k \Rightarrow q_{k+1}$

} proof induction

$$q_0 \Rightarrow q_1 \Rightarrow q_2 \Rightarrow q_3 \Rightarrow \cdots \Rightarrow$$

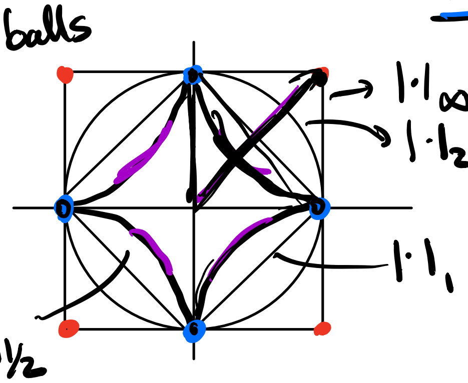
Ex. finding a counterexample:

consider the p-"norm" for $p = \frac{1}{2}$ $\quad |x|_{1/2} = \left(\sum_i |x_i|^{1/2}\right)^2$

show this $\frac{1}{2}$-"norm" doesn't satisfy the triangle inequality

WTS: $\exists\, x, y$ s.t. $|x+y|_{1/2} \nleq |x|_{1/2} + |y|_{1/2}$ would need to be true for $\forall x, y$

2D $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad$ unit balls



$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ an counter example

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

BACK TO LINEAR ALGEBRA: BASES

vector space $\mathcal{V}$

set of vectors $\{v_i\}_{i=1}^{n}$

linear combinations: $v_1 x_1 + \cdots + v_n x_n = \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{coeffs.}}$

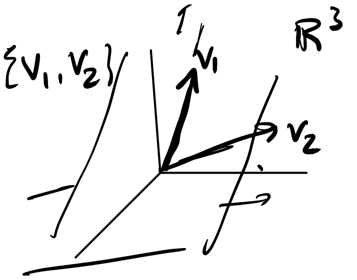
span of $\{v_i\}_{i=1}^{n}$: $\{y \mid y = Vx, x \in \mathbb{R}^n\}$

vector $w \in \mathcal{V}$

is linear dep on $\{v_i\}_{i=1}^{n}$

if $w \in$ span of $\{v_i\}_{i=1}^{n}$

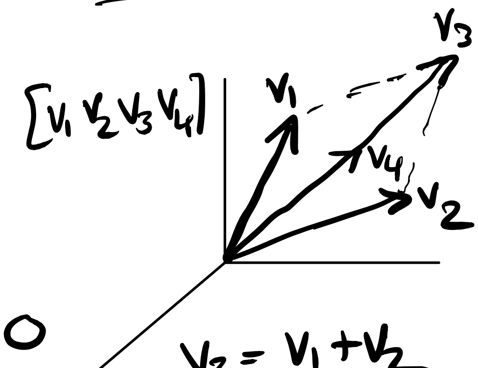
$\Rightarrow \exists x \in \mathbb{R}^n$ s.t. $w = Vx$



a set of vectors $\{v_i\}_{i=1}^{n}$

is linear dep if one v_i is lin dep on the others

$\Rightarrow \exists x \in \mathbb{R}^n, x \neq 0$ s.t. $Vx = 0$



$v_3 = v_1 + v_2$

$v_4 = \frac{1}{2} v_3$

$v_4 = \frac{1}{2} v_1 + \frac{1}{2} v_2$

a set of vectors $\{v_i\}_{i=1}^{n}$

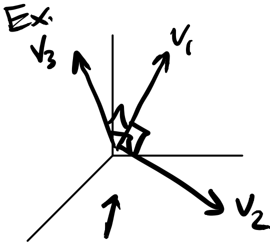
is linearly independent

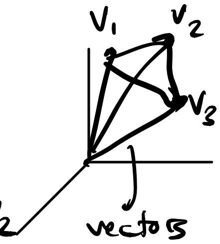
if none of the vectors are lin dep on ea. other

$\Rightarrow$ if $Vx = 0 \Rightarrow x = 0$

$x \in \mathbb{R}^n$

Ex.

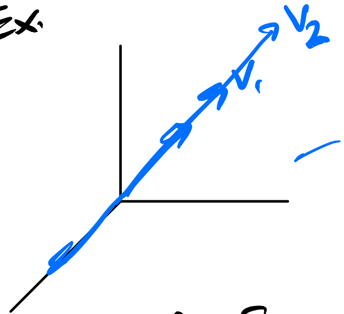




vectors not on same plane

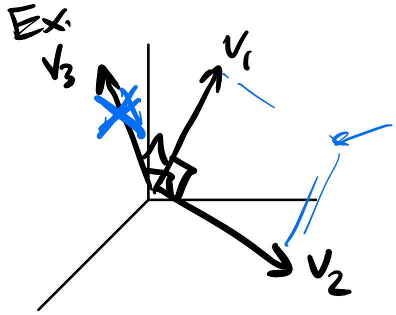
Proposition: if $[v_1 \cdots v_k \, v_{k+1}]$ lin ind. $\Rightarrow [v_1 \cdots v_k]$ lin ind.

Ex.



$[v_1, v_2, \cancel{v_3}]$

Ex.



WTS: if $[v_1 \cdots v_k \, v_{k+1}]$ lin ind $\Rightarrow [v_1 \cdots v_k]$ lin ind.

ASSUME NOT: $[v_1 \cdots v_k]$ lin dep.

$$\Rightarrow \exists x \neq 0 \text{ s.t. } [v_1 \cdots v_k]\, x = 0 \quad x \in \mathbb{R}^k$$

$$\Rightarrow \overbrace{[v_1 \cdots v_k \, v_{k+1}]}^{k+1} \begin{bmatrix} x \\ 0 \end{bmatrix} = \underbrace{[v_1 \cdots v_k]\, x}_{0} + \cancel{v_{k+1}}\, 0$$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \neq 0$$

$$\Rightarrow \underbrace{[v_1 \cdots v_k \, v_{k+1}]}_{\text{lin dep}} \quad \times$$

proving contrapositive...

if $p \Rightarrow q$ statement

$\neg q \Rightarrow \neg p$ contrapositive

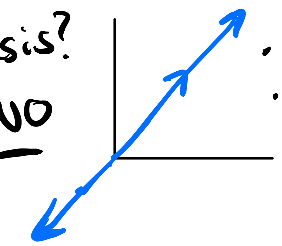
BASIS – a set of vectors $\{v_i\}_{i=1}^{n} \subset V$

- $\{v_i\}_{i=1}^{n}$ span all of V "generates V"
- $\{v_i\}_{i=1}^{n}$ linearly independent

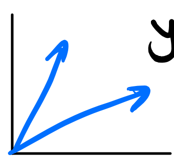
Ex.

Basis?

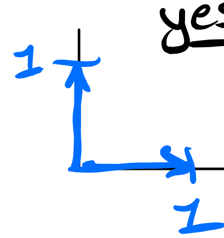
NO



· lin dep
· don't span



yes



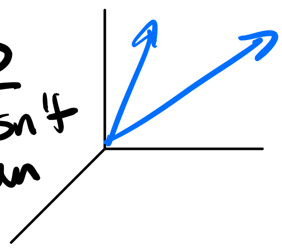
yes

standard basis in $\mathbb{R}^2$

$\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$

NO

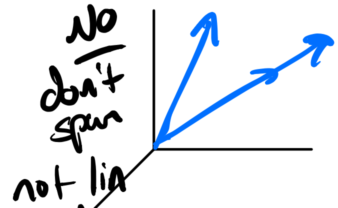
doesn't span

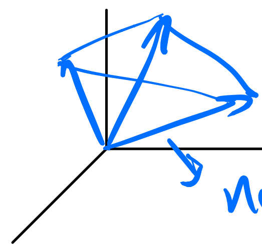


NO

don't span

not lin ind.



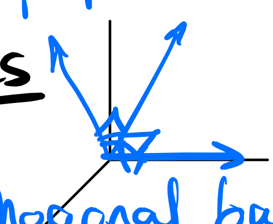


yes

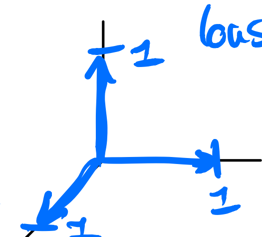
not orthonormal

all perpendicular

yes



standard basis in $\mathbb{R}^3$



$$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$$

orthogonal basis

if also length 1....

orthonormal → a $\boxed{\text{rotation}}$ converts your basis into the standard basis

$$y = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{\text{coeff of } y} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{coords}}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$