BASES, COORDINATES,
INVERSES, SIMILARITY TRANSFORMS

vector space V $v_i \in V$

$\{v_i\}_{i=1}^n$ is basis

- lin ind
- span all of V

$\{v_i\}_{i=1}^n$ is a basis for V

FUNDAMENTAL FACT:

any lin ind set of vectors has fewer (or equal) vectors than any spanning set

$\underbrace{\{v_i\}_{i=1}^m}_{\text{lin ind.}}$ $v_i \in V$ $\boxed{m \le n}$ $\underbrace{\{w_i\}_{i=1}^n}_{\text{spanning}}$ $w_i \in V$

Completing a basis

can always add elements w_i to $\{v_i\}_{i=1}^m$ until $\underbrace{\{v_i\}_{i=1}^m \cup \{w_i\}}_{\text{basis}} \subset \{w_i\}_{i=1}^n$

Steinitz Exchange LEMMA

BASIS:

- all bases have the same # of elements

Note: # of elements in a set = <u>cardinality</u>

[cardinality of a basis for V = <u>dimension</u> of V

standard basis for $\mathbb{R}^n$

cols of $I = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{bmatrix}$

- "largest" (cardinality) lin ind. set
- "smallest (cardinality) spanning set

<u>Coordinates</u>: representations of vectors wrt a basis.

$$P = [\underbrace{P_1 \cdots P_n}_{\text{form a basis for } \mathbb{R}^n}] \qquad \underline{x} = \underline{P}x' = [P_1 \cdots P_n]\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

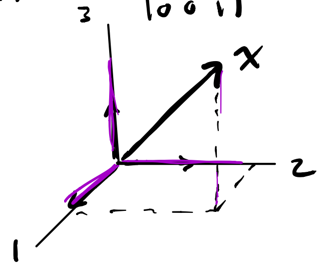
$$= \underline{P_1}x_1' + \cdots + \underline{P_n}x_n'$$

"<u>coords</u> <u>are coeffs</u>" $\quad x'$ is the coordinates of x wrt (the cols of) P

Ex. standard basis

$\mathbb{R}^3$: $P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad x = Px'$

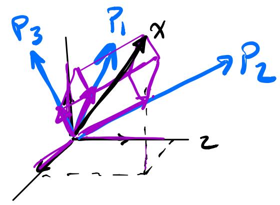
$= Ix'$



$x = x' = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Ex

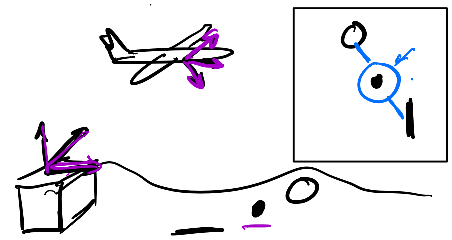
$P=[P_1 P_2 P_3]$



$x = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}\begin{bmatrix}x'_1\\x'_2\\x'_3\end{bmatrix}$ $\quad x'=\begin{bmatrix}1/2\\1/2\\1/4\end{bmatrix}$

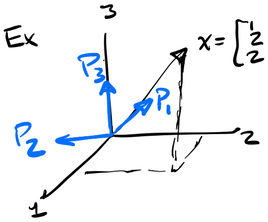
Ex. Aero

camera on a drone



Construct $\bar{x}$ as lin comb P_i's

Ex



$x=\begin{bmatrix}1\\2\\2\end{bmatrix}$ $\quad P=\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix} \end{array} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$

$x = Px'$

✓ $\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1\\-2\\2\end{bmatrix}$

Basis cols of P:

- cols span the whole space "enough directions"
- lin ind. → "no redundant directions"

⇒ matrix P is invertible $\quad x = Px'$

"switch back & forth between x' coords & x (coords) without losing information

$$P^{-1}x = P^{-1}Px'$$
$$P^{-1}x = x'$$

cols of P

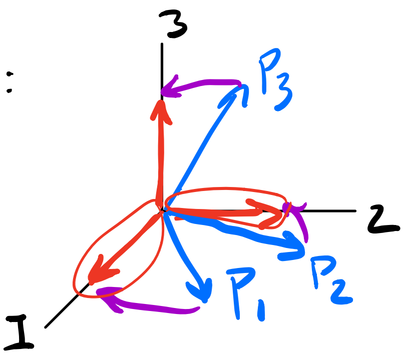
- span $\mathbb{R}^n$ → "every x has an x'" – P is onto — P is surjective — "can reach anywhere in the space"
- lin ind. → ea. x has a unique x' — P is one-to-one — P is injective — no redundant coords

if P is both injective & surjective = one-to-one onto = bijective = invertible

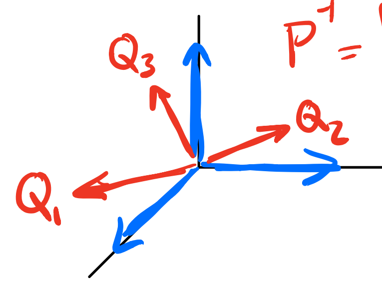
"reaches anywhere w/ unique coords"

For matrices $P^{-1}P = I = PP^{-1} = I$

BRIEF PICTURE:



cols of P^{-1}?



$P^{-1} = [Q_1 Q_2 Q_3]$

FACTS ABOUT INVERSES

$P \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$ P invertible

Properties

- $(P^{-1})^{-1} = P$
- $(kP)^{-1} = \frac{1}{k} P^{-1}$
- $(PQ)^{-1} = Q^{-1}P^{-1}$ $Q \in \mathbb{C}^{n \times n}$
- $\det(P^{-1}) = \frac{1}{\det(P)}$
- $P^{-1} = \frac{1}{\det(P)} \mathrm{Adj}(P)$ → matrix of cofactors

  det of sub matrices "messy"

Equivalent Properties P square

All of these statements are equivalent

- P is invertible ie P^{-1} exists
- row reduce P to I
- col reduce P to I
- P is a product of elementary matrices

  } → Gaussian elimination (solve a system of eqns)
- P(square) and full row rank
- P(square) and full col rank
- cols of P are lin ind. (P square)
- rows of P are lin ind. (P square)
- $y = Px$ has a unique soln for ea y

Wikipedia

- P has a trivial nullspace, $\text{null}(P) = \{0\}$
- $Px = 0 \Rightarrow x = 0$
- cols form a basis
- P^T is invertible
- 0 is not an eigenvalue of P
- $\det(P) \neq 0$ ← "no dim collapse"
- $\exists\, Q$ s.t. $PQ = QP = I$ $(P^{-1} = Q)$
- P has a left & right inverse

Computational Inverse Facts

- 2x2 inverse

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad P^{-1} = \frac{1}{\det P}\,\text{Adj}(P) = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

only true in 2x2 case

$$P^{-1} = \frac{1}{\det(P)}\left[\text{Tr}(P)\,I - P\right]$$

- 3x3

$$P^{-1} = \frac{1}{\det(P)}\left[\frac{1}{2}\left[\text{tr}(P)^2 - \text{tr}(P^2)\right]I - P\,\text{tr}(P) + P^2\right]$$

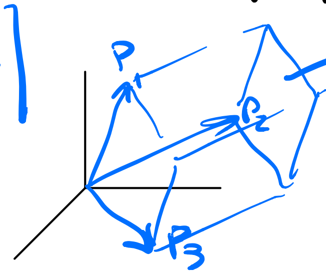
- $n \times n$ similar formulas ...

Side Note:

tr(·): sum of diagonal

det: (signed) volume of the parallelpipped given by the cols of matrix

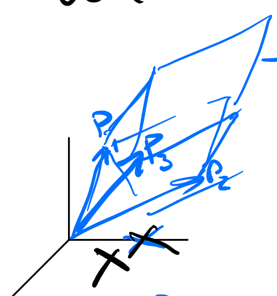
$P = [P_1 \, P_2 \, P_3]$



signed volume = det

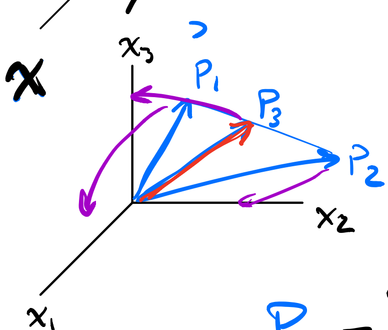
"determinant becomes negative if you flip the volume inside out"

if $\det(P) = 0$



squashed parallel pipped "no volume"

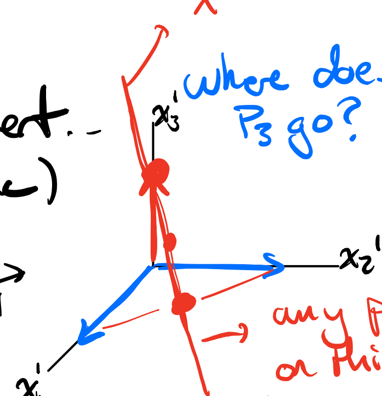
$x = Px'$



P

try to invert... (impossible)

P^{-1}



where does P_3 go?

any point on this line gets mapped to P_3 when you apply P

$P_3 = Px'$

$x' = P^{-1} P_3$

Block Matrix Inversion:

$$P^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \star$$

$$= \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix} \star\star$$

Caveat:

- D^{-1} exist and $(A-BD^{-1}C)^{-1}$ exist OR
- A^{-1} exist and $(D-CA^{-1}B)^{-1}$ exist

$$\left.\begin{matrix} (A-BD^{-1}C)^{-1} \\ (D-CA^{-1}B)^{-1} \end{matrix}\right\} \to \text{Schur complements}$$

SOURCE: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \star$

$$= \underbrace{\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}}_{\text{easy to invert}} \begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \star\star$$

$$P = \begin{array}{c|cc} & n_1 & n_2 \\ \hline n_1 & A & B \\ n_2 & C & D \end{array}, \quad P \in \mathbb{R}^{n\times n}$$

$n_1 = n_2$ not necessary

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Woodbury Matrix Identity
(specific case Sherman Morrison Formula)

$A \in \mathbb{R}^{n \times n}$

in general $(A+B)^{-1} \neq A^{-1} + B^{-1}$

scalar $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$

$$(\underbrace{A}_{\text{matrix}} + \underbrace{UCV}_{\text{add}})^{-1} = \underbrace{A^{-1}}_{\text{inverse}} - A^{-1}U\underbrace{(C^{-1} + VA^{-1}U)^{-1}}_{\text{add to inverse}}VA^{-1} \quad \star$$

U, C, V dimensions to match — UCV needs to be $n \times n$

A^{-1} needs to exist and $(C^{-1} + VA^{-1}U)^{-1}$ exists

$$\left(\begin{bmatrix} A \end{bmatrix} + \underbrace{\begin{bmatrix} U \end{bmatrix}\begin{bmatrix} C \end{bmatrix}\begin{bmatrix} V \end{bmatrix}}_{\text{addition is low rank}}\right)^{-1} = A^{-1} - \underbrace{\begin{bmatrix} \; \end{bmatrix}}_{A^{-1}U}\underbrace{\begin{bmatrix} \; \end{bmatrix}}_{(C^{-1}+VA^{-1}U)^{-1}}\underbrace{\begin{bmatrix} \; \end{bmatrix}}_{VA^{-1}}$$

$\boxed{\text{extreme case}}$

Sherman Morrison Formula
- U col vector
- V row vector
- C scalar

$\rightarrow \left(\begin{bmatrix} C^{-1} \end{bmatrix} + \begin{bmatrix} V \end{bmatrix}\begin{bmatrix} A^{-1} \end{bmatrix}\begin{bmatrix} U \end{bmatrix}\right)^{-1}$ scalar

trick behind computationally efficient Kalman filter

$$\underbrace{(A + UCV)^{-1}}_{\text{new covariance}}$$

covariance (A), Kalman gain (C), new measurement (V)

Neumann series

if $\lim_{n\to\infty} (I-A)^n = 0 \implies A^{-1} = \sum_{n=0}^{\infty} (I-A)^n$

not on a test   matrix version of harmonic series

Derivative of Inverse:

$P(t)$ $\boxed{\frac{dP^{-1}}{dt} = -P^{-1}\frac{dP}{dt}P^{-1}}$ ←

$$\frac{d}{dt}P^{-1}P = \underbrace{\frac{d}{dt}I}_{0} \implies \underbrace{\frac{dP^{-1}}{dt}P + P^{-1}\frac{dP}{dt} = 0}$$

$$\implies \frac{dP^{-1}}{dt} = -P^{-1}\frac{dP}{dt}P^{-1}$$

Elementary Matrices & computing inverses next time.

Bases for functions:

standard basis: $\underline{\delta(t)}$ t index. coordinates: $\underline{f(t)}$

Fourier basis: $\cos(n\omega t)$, $\sin(n\omega)$ coordinates: $F(\omega)$

P P^{-1}

$\to \mathcal{F}(\cdot)$ $\mathcal{F}^{-1}(\cdot)$