How do we compute an inverse?

$$y = Px \qquad P \text{ invertible} \qquad P \in \mathbb{R}^{n\times n}$$

- n equations
- n unknowns

1. $P^{-1}y = P^{-1}Px$

$$P^{-1}y = x$$

2. GAUSSIAN ELIMINATION (ROW REDUCTION)

RR operations.

- swapping rows
- × scalar
- + rows

$\Big\}\rightarrow$ these operations can be done by left multiplying by a matrix.

swapping rows...

$E_i P \Rightarrow$ swapping 2 rows

$$E_i = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & 1 & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{bmatrix}$$

$$E_iP = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & 0 & 1 & \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} - & p_1^T & - \\ - & p_i^T & - \\ - & p_j^T & - \\ - & p_n^T & - \end{bmatrix} = \begin{bmatrix} - & p_1^T & - \\ - & p_j^T & - \\ - & p_i^T & - \\ - & p_n^T & - \end{bmatrix}$$

E_i : elementary matrices

scaling row... by α

$$E_i = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \ddots \; 1 \end{bmatrix}$$

adding rows ...

$$E_i = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ & 1 & 0 \cdots 1 & \cdots & 0 \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

breaking solving lin sys. or computing an inverse into multiplication by these matrices

$y = Px \rightarrow$ solve for x.

$$[P \mid x] \longrightarrow E_k \cdots E_2 E_1 [P \mid y] = [\underbrace{E_k \cdots E_1 P}_{I} \mid \underbrace{E_k \cdots E_1 y}_{x}]$$

solve for all x's at once... by row reducing. (x solved for x.)

$[P \mid I]$ ← cols are standard basis vectors → solving for every y all at once...

$$[\underbrace{E_k \cdots E_1 P}_{I} \mid \underbrace{E_k \cdots E_1}_{P^{-1}}]$$

$$\boxed{P^{-1} = E_k \cdots E_1}$$

instead of row reducing → can col reduce...

$$\left[\frac{P}{I}\right] \xrightarrow{\text{col red.}} \left[\frac{P E_k \cdots E_1}{E_k \cdots E_1}\right] \begin{matrix} \rightarrow I \\ \rightarrow P^{-1} \end{matrix}$$

Note: solving for all y's at once...

ex. $y = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Px$ $\quad \left[\underline{P} \,\middle|\, \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right]$ $\quad y = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} y_1 + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} y_n$

$y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = Px$ $\quad \left[P \,\middle|\, \begin{matrix} 0 \\ 1 \\ \vdots \\ 0 \end{matrix}\right]$

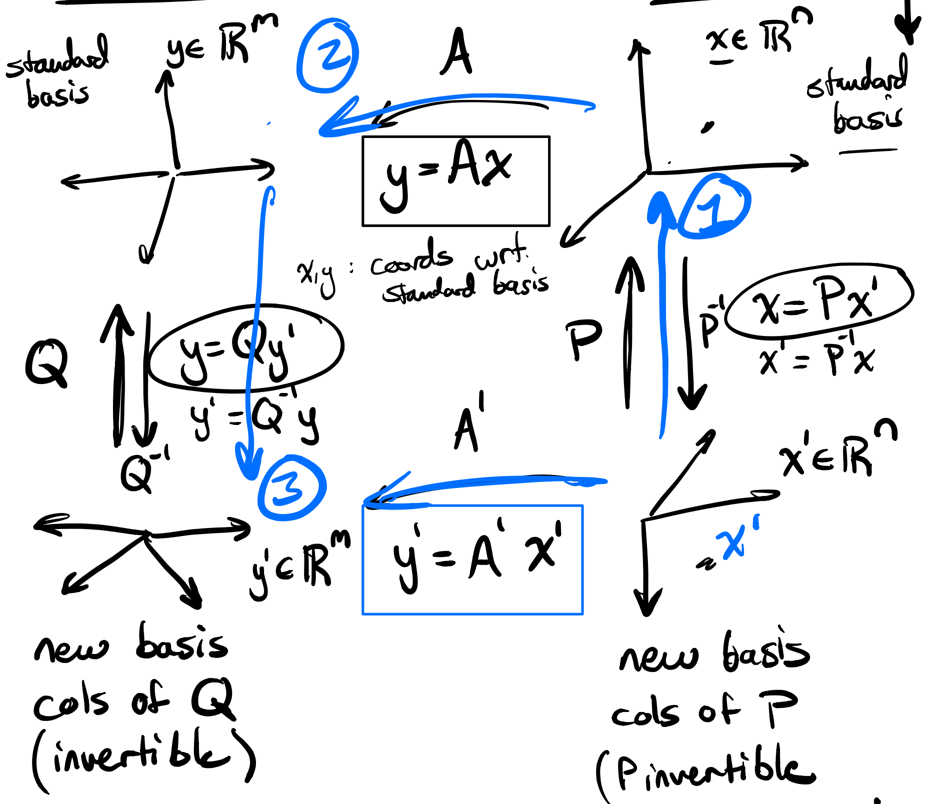
$E_i\, [\underline{P} \mid \mid \mid \mid \mid]$ $\quad [P \mid I]$ $\quad y = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

$\rightarrow \left[E_i P \,\middle|\, E_i \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}, E_i \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}\right]$ $\quad y =$

Similarity Transforms:

"coordinate transformations applied to a matrix"

CODOMAIN **DOMAIN** → for function $f(x) = Ax$



$$y = Ax$$
$$Qy' = APx'$$
$$y' = \underbrace{Q^{-1}AP}\, x'$$
$$\boxed{A' = Q^{-1}AP}$$

$\Leftrightarrow$

$$y' = A'x'$$
$$Q^{-1}y = A'P^{-1}x$$
$$y = QA'P^{-1}x$$
$$\boxed{A = QA'P^{-1}}$$

$$A'x' = Q^{-1}AP\,x'$$

1

2

3

For square A's...

$$\begin{tikzcd}
y \in \mathbb{R}^n & & x \in \mathbb{R}^n \arrow[ll, "A \in \mathbb{R}^{n \times n}"'] \\
y' \in \mathbb{R}^n \arrow[u, "y = Py'"] & & x' \in \mathbb{R}^n \arrow[ll, "A'"'] \arrow[u, "x = Px'"']
\end{tikzcd}$$

$\underbrace{\boxed{A' = P^{-1}AP}}_{\text{similarity transform}}$ $\rightarrow$ say that A & A' are <u>similar</u>

$A' = P^{-1}AP$

$\det(A') = \det(P^{-1}AP)$

$= \cancel{\det(P^{-1})}\det(A)\cancel{\det(P)}$

Similar...

- <u>same</u> eigen<u>values</u> (different eigenvectors)
- $\det(A) = \det(A')$ $\rightarrow$ $\lambda_1 \cdots \lambda_n$ eigenvalues of A $\leftarrow$
  $\det(A) = \lambda_1 \cdots \lambda_n$
  $tr(A) = \lambda_1 + \cdots + \lambda_n$
- $tr(A) = tr(A')$

$tr(A') = tr(P^{-1}AP)$

$= tr(\underbrace{PP^{-1}}_{I}A)$

$= tr(A)$

<u>Determinant Properties</u>

- $\det(I) = 1$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- for square A & B.
  $\det(AB) = \det(A)\det(B)$
- $A \in \mathbb{R}^{n \times n}$
  $\det(cA) = c^n \det(A)$
- $\det(A) = \lambda_1 \cdots \lambda_n$
- $\underbrace{A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}}_{\text{diagonal}}$ $\underbrace{\det(A) = a_1 \cdots a_n}_{\text{vol interpretation}}$

<u>Trace Properties</u>

- $tr(A+B) = tr(A) + tr(B)$
- $tr(cA) = c\,tr(A)$
- $tr(A) = tr(A^T)$
- $tr(AB) = tr(BA)$

Orthonormal Coordinate Transformation:

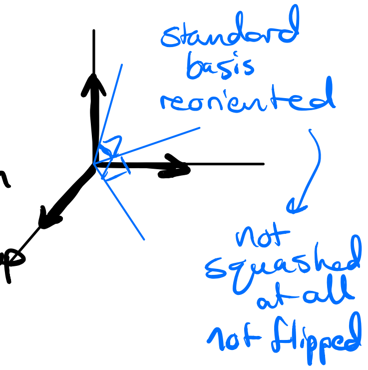
"ortho" — orthogonal

"normal" — norm = 1

orthonormal matrix has cols (and rows) that are orthogonal to ea. other & mag. 1.

$$R = \begin{bmatrix} | & & | \\ R_1 & \cdots & R_n \\ | & & | \end{bmatrix} \qquad R = \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix}$$

$R \in \mathbb{R}^{n \times n}$ "orthogonal (rotation) matrix"



R_1, R_2 rotations $\Rightarrow$ $R_1 R_2$ rotation

SO(n) : special orthogonal group

$\mathcal{U} \in \mathbb{C}^{n \times n}$ "unitary matrix"

$\mathcal{U}_1, \mathcal{U}_2$ unitary $\Rightarrow$ $\mathcal{U}_1 \mathcal{U}_2$ unitary

SU(n) : special unitary group

without det cond — rotations & reflections

$$\mathrm{SO}(n): \; R \in \mathbb{R}^{n \times n} \text{ s.t. } \boxed{R^T R = I}, \; \boxed{\det(R) = 1}$$

$$\mathrm{SU}(n): \; \mathcal{U} \in \mathbb{C}^{n \times n} \text{ s.t. } \boxed{\mathcal{U}^* \mathcal{U} = I} \; \boxed{\det(\mathcal{U}) = 1}$$

"rotations"

→ rotations

SU(n) → "spin in quantum mechanics"

SU(2) Pauli matrices

$R = [R_1 \cdots R_n]$ cols are <u>orthogonal</u> & <u>norm 1</u>.

- $\underline{R_i^T R_j = 0}$ $\quad i \neq j$ orthogonal
- $\underline{R_i^T R_i} = 1$ $\quad$ norm 1.

$R_i^T R_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\longrightarrow R^T R = I$

$$\boxed{R^{-1} = R^T}$$

Show that $R^T R = I$:

$$\begin{bmatrix} \cdots R_1^T \cdots \\ \cdots R_n^T \cdots \end{bmatrix} \begin{bmatrix} R_1 \cdots R_n \end{bmatrix} = \begin{bmatrix} R_1^T R_1 & \cdots & R_1^T R_n \\ \vdots & \ddots & \vdots \\ R_n^T R_1 & \cdots & R_n^T R_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & 1 \end{bmatrix}}_{I}$$

<u>Useful</u> for <u>two reasons</u>:

- $R^{-1} = R^T$: inverse simple to compute.
- applying R to a space does not change metric properties or "<u>measurements</u>"
  - relative angles and lengths

based on inner product

R is an iso<u>metry</u>

why?

$x = \underline{R}x'$, $y = \underline{R}y'$

$$\underline{x^T y} = (x')^T \underbrace{R^T R}_{I} y' = \underline{(x')^T y'}$$

$\sqrt{x^T x}$ $\quad x^T y = |x||y| \cos\theta$ $\Big\} \longrightarrow$ relative angles & lengths don't change under rotations

2x2 Rotation $R \in \mathbb{R}^{2\times 2}$

$$R = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

counterclockwise rotation

useful.

Can check $R^T R = I$, $\det(R) = 1$

3D Rotations:

- Euler angle approach → has some problems gimbal lock
- Quaternions 4D complex #'s }

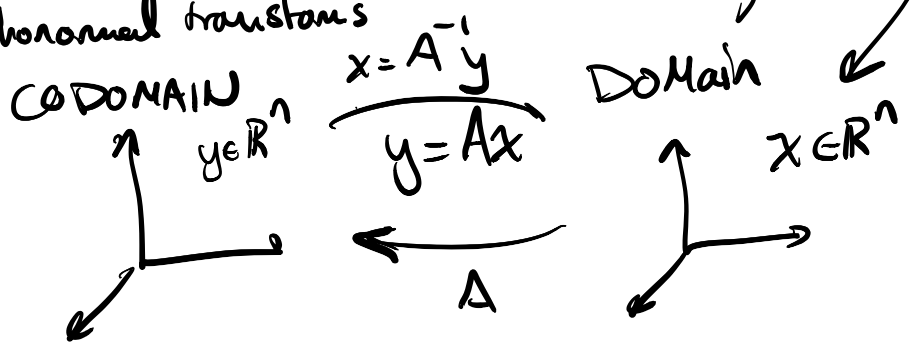
SO(3) .... Quaternions .... SU(2)

richer

Recap:

- bases
- coordinate transforms } → invertible square matrices
- inverses
- similarity transforms
- orthonormal transforms

→ square invertible matrices



what about non-square matrices?

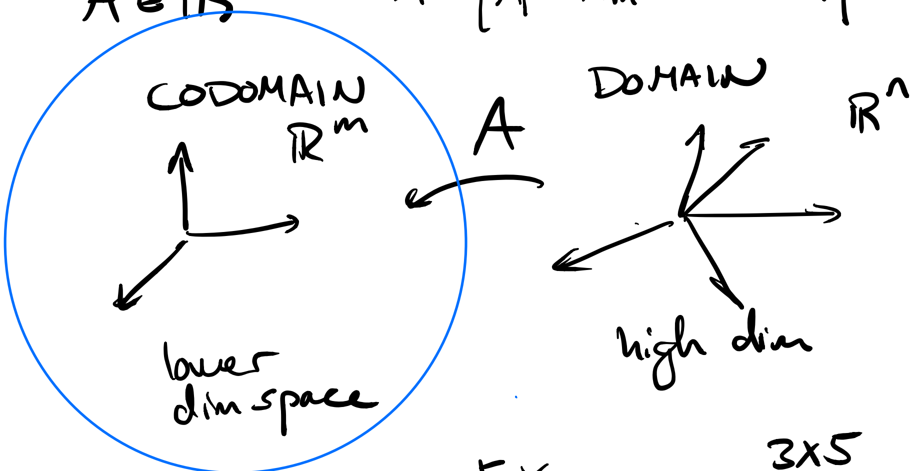
⇒ DOMAIN & CODOMAIN have different dimensions

PREVIEW: Range(A) := R(A)

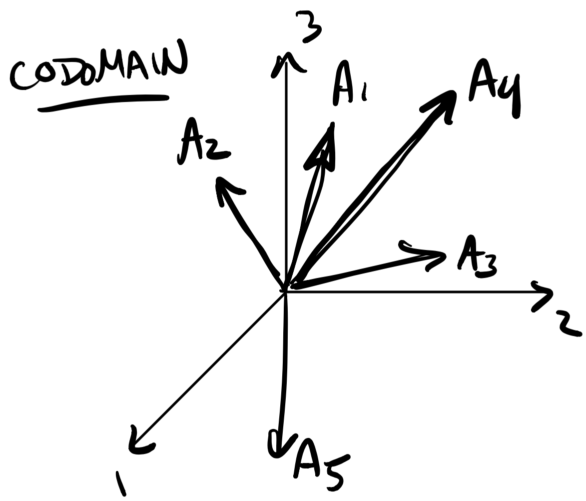
Nullspace(A) := N(A)

FAT MATRICES "more cols than rows"

$A \in \mathbb{R}^{m \times n}$ $A = [A_1 \cdots A_m \; A_{m+1} \cdots A_n]$



Ex. $A \in \mathbb{R}^{3 \times 5}$ $A = [A_1 \cdots A_5]$



if I said "use the cols of A to write a basis for $\mathbb{R}^3$"

↳ redundant or extra columns

solve $y = Ax$ $\quad y \in \mathbb{R}^3,\ x \in \mathbb{R}^5$

by row reducing

$$[A \mid y] \sim [\underbrace{I}\ \underbrace{A}_{\text{extra cols.}} \mid \tilde{y}]$$

assume lin ind.

$$A = [\underbrace{A_1 A_2 A_3}_{B} \mid \underbrace{A_4 A_5}_{C}]$$

$B \in \mathbb{R}^{3\times 3}$ $\quad C \in \mathbb{R}^{2\times 3}$

general case

$A \in \mathbb{R}^{m\times n}$

$A = [B \mid C]$

$B \in \mathbb{R}^{m\times m}$ $\quad C \in \mathbb{R}^{m\times(n-m)}$

if cols of B are lin ind.

$\Rightarrow$ B^{-1} exists...

$$B^{-1}A = [B^{-1}B \mid B^{-1}C] = [I \mid B^{-1}C]$$

consider $\boxed{N = \begin{bmatrix} B^{-1}C \\ -I \end{bmatrix}}$ $\quad N \in \mathbb{R}^{5\times 2}$

general case

$N \in \mathbb{R}^{n\times(n-m)}$

$$AN = [B\ C]\begin{bmatrix} B^{-1}C \\ -I \end{bmatrix} = BB^{-1}C - C = 0$$

matrix

$AN = 0$

$N = [N_1\ N_2]$

$AN_1 = 0 \quad AN_2 = 0$

both N_1 & N_2 are in the <u>nullspace</u> of <u>A</u>

Nullspace of A.
(kernel)

$\mathcal{N}(A) = \{ \underline{x} \mid Ax = 0,\ x \in \mathbb{R}^n \} \subseteq \text{DOMAIN } (\mathbb{R}^n)$

if $A \in \mathbb{R}^{m \times n}$ $A = [B \mid C]$ $B \in \mathbb{R}^{m \times m}$ invertible $C \in \mathbb{R}^{m \times (n-m)}$

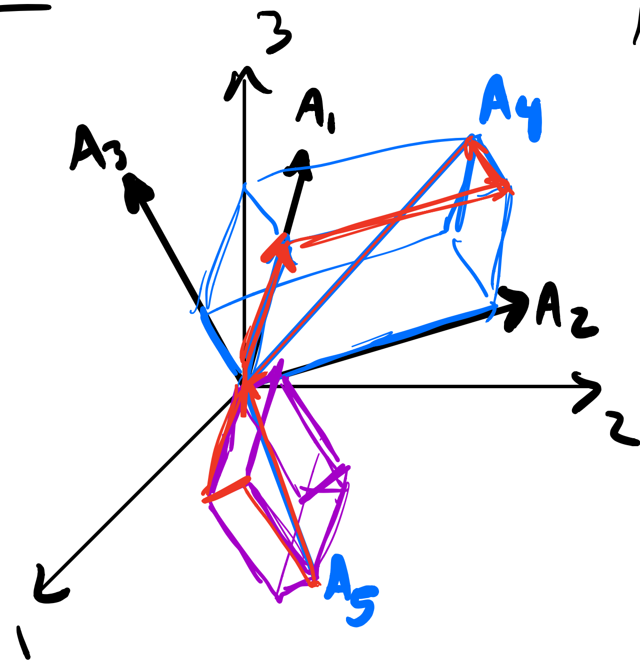
$\Rightarrow$ $\boxed{\text{span of cols of } N = \begin{bmatrix} B^{-1}C \\ -I \end{bmatrix} \text{ is nullspace of } A}$

constructing basis for nullspace

Nullspace: coeffs of lin combs of cols of A that end up at 0.

if $x \in N(A)$ $Ax = [A_1 \cdots A_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A_1 \underline{x_1} + \cdots + A_n \underline{x_n}$
$= 0$

<u>CODOMAIN</u>



$A = [B\ C] = [A_1 A_2 A_3 \mid A_4 A_5]$

$N = \begin{bmatrix} B^{-1}C \\ -I \end{bmatrix} = [N_1\ N_2]$

$B^{-1}C = [\underline{B^{-1}A_4}\ \underline{B^{-1}A_5}]$
coeffs of A_4 wrt the basis given by cols of B

$B = [A_1 A_2 A_3]$

$$A_4 = [\underbrace{A_1 A_2 A_3}_{B}]u \implies u = B^{-1} A_4$$

$$A_5 = [\underbrace{A_1 A_2 A_3}_{B}]v \implies v = B^{-1} A_5$$

$$A = [B\ C]N_1 = Bu - A_4 = BB^{-1}A_4 - A_4 = 0$$

$$A = [B\ C]N_2 = Bv - A_5 = BB^{-1}A_5 - A_5 = 0$$

sets of nonzero coeffs $\longrightarrow$ end up at 0