TOPICS: Nullspace, Range, rank (dims)
rank-nullity thm, fund thm of lin alg.
LS, minimum norm solutions

NULLSPACES: (FAT MATRIX) $A \in \mathbb{R}^{m\times n}$

$A = [A_1 \cdots A_n]$ only k lin ind cols.

$B = [\underbrace{A_1 \cdots A_k}_{\text{lin ind.}}]$

$$A = m\left[\begin{array}{cc} \overset{k}{{}_m B} & \overset{k}{{}_m B}\,\overset{n-k}{{}_k D} \end{array}\right]$$

$A = [B \;\; B\overset{n}{D}]$

all other cols can be written as lin combs of cols of B

cols of D are the lin combs of the cols of B that give $[A_{k+1} \cdots A_n]$

last time: assumed B was invertible → not necessary

$$\underbrace{[A_1 \cdots A_k]}_{B}[D_{k+1} \cdots D_n] = [A_{k+1} \cdots A_n]$$

$$[BD_{k+1} \cdots BD_n] = [A_{k+1} \cdots A_n]$$

$$AN = [B \mid BD]\begin{bmatrix} D \\ -I \end{bmatrix} = 0$$

DIRECT CONSTRUCTION of nullspace of A

$$BD - BD = 0$$

$n-k$ cols

$A = [\underbrace{A_1 \cdots A_k}_{k} \; \underbrace{A_{k+1} \cdots A_n}_{n-k}]$

$N = \begin{bmatrix} D_{k+1} \cdots D_n \\ -I \end{bmatrix}$

(*) DIRECT PROOF OF RANK NULLITY Thm

Ex.

$A = [A_1 \cdots A_5]$

$= [\underbrace{A_1 A_2}_{B} \mid A_3 A_4 A_5]$ $\longrightarrow [A_3 A_4 A_5] = \underbrace{[A_1 A_2]}_{B} \underbrace{[D_3 D_4 D_5]}_{D}$

$A_3 = BD_3, \quad A_4 = BD_4, \quad A_5 = BD_5$

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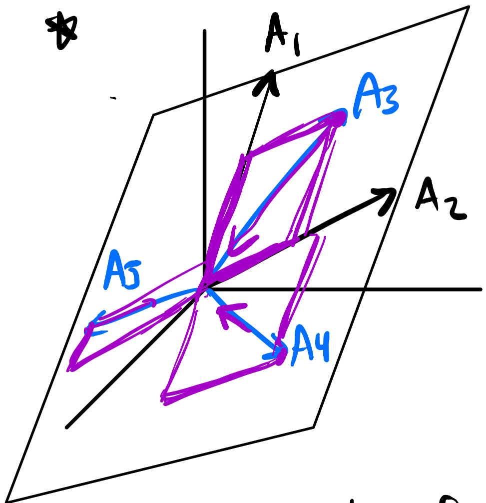


$$N = \underbrace{\begin{bmatrix} D \\ -I \end{bmatrix}}_{} = \begin{bmatrix} D_3 & D_4 & D_5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

coords of 0 relative to A

PROP: the cols of N are lin ind.

try to break . . . $Nz = 0 \implies z = 0$

$$\begin{array}{c} 3 \\ \begin{bmatrix} D_3 & D_4 & D_5 \\ \hline -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ I \end{array} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} * \\ * \\ \hline -z_1 \\ -z_2 \\ -z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} z_1 &= 0 \\ z_2 &= 0 \\ z_3 &= 0 \end{aligned}$$

"

$$\begin{bmatrix} D \\ -I \end{bmatrix} z = \begin{bmatrix} Dz \\ z \end{bmatrix} \to 0$$

lin ind: if $Nz = 0$ then $z = 0$

"the only lin comb of vectors that gives you 0 is the coeffs being all 0"

you can prove lin ind of a set of cols by proving lin ind for only a few rows

$N_1 z_1 \cdots N_3 z_3 = 0$

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \quad \text{if } N_2 z = 0 \implies z = 0$$

cols of N_2 are lin ind.

$\implies$ cols of N are lin ind.

Nullspace Construction

$A = [B \ \ BD]$ — full set of lin ind cols.

$$N = \begin{bmatrix} D \\ -I \end{bmatrix}$$

cols of N are lin ind.

$AN = 0$

col perspective on nullspace

Nullspace: $A \in \mathbb{R}^{m \times n}$

$$N(A) = \{ x \mid Ax = 0,\ x \in \mathbb{R}^n \} = \text{span of cols of } N$$

Row perspective.

$$A = \begin{bmatrix} - & \bar{a}_1^T & - \\ & \vdots & \\ - & \bar{a}_m^T & - \end{bmatrix}$$

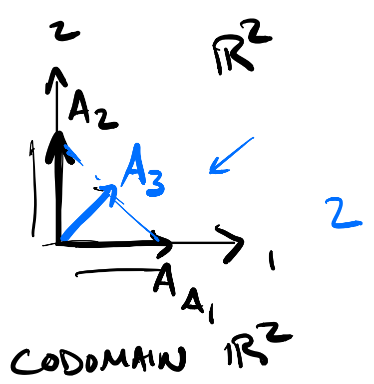
break up A into rows.

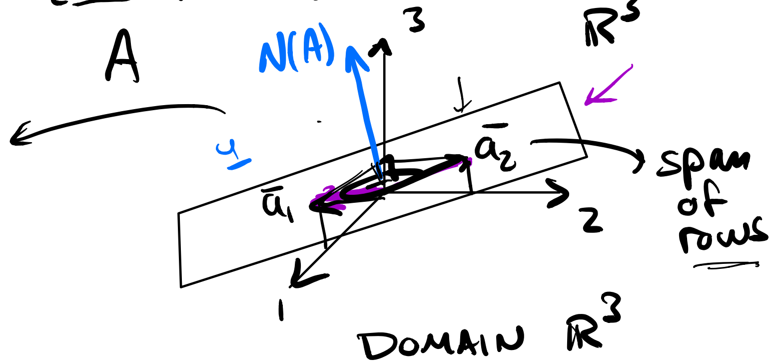
$$Ax = \begin{bmatrix} - & \bar{a}_1^T & - \\ & \vdots & \\ - & \bar{a}_m^T & - \end{bmatrix} x = \begin{bmatrix} \bar{a}_1^T x \\ \vdots \\ \bar{a}_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

"the nullspace of A is orthogonal to the rows of A"

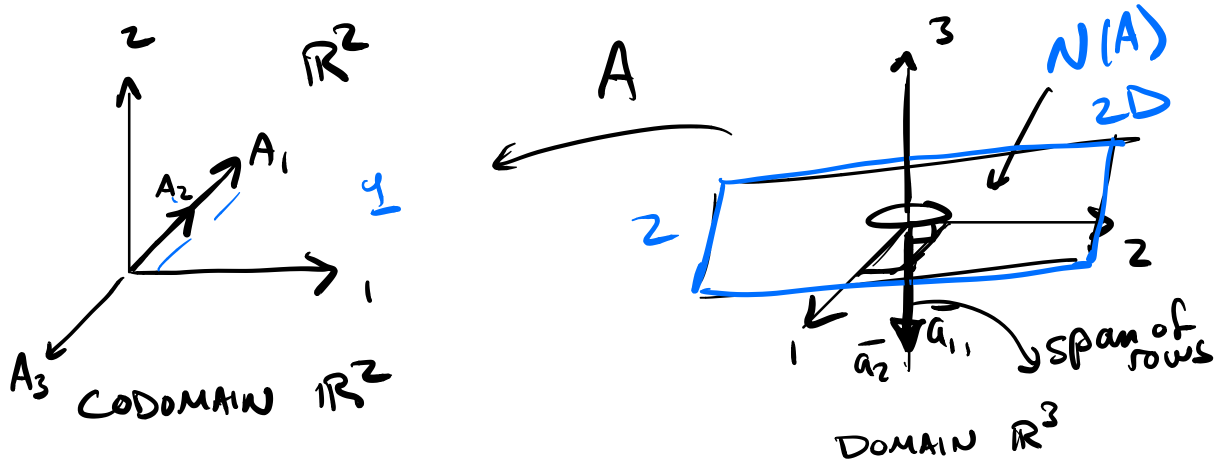
Ex. $A \in \mathbb{R}^{2 \times 3}$

$$A = [A_1 A_2 A_3] = \begin{bmatrix} - & \bar{a}_1^T & - \\ - & \bar{a}_2^T & - \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$





Ex. $A \in \mathbb{R}^{2\times 3}$ $\quad A = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} = \begin{bmatrix} — \bar{a}_1^T — \\ — \bar{a}_2^T — \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -1 \\ 1 & 1/2 & -1 \end{bmatrix}$



Lose dim in span of cols $\Rightarrow$ gain a dim in the nullspace

Rank Nullity: $A \in \mathbb{R}^{m\times n}$ $\quad$ Proof $\Uparrow$ ★

$$\underbrace{\dim(R(A))}_{k} + \underbrace{\dim(N(A))}_{n-k} = n$$

Span of cols of A = Range of A = $R(A)$

Range of A:

$$R(A) = \{y \mid y = Ax,\ x \in \mathbb{R}^n\} = \text{span of cols of } A$$

Rank: $A \in \mathbb{R}^{m \times n}$

col rank $= \dim(R(A)) =$ # of lin ind. cols

row rank $= \dim(R(A^T)) =$ # of lin ind. rows

Prop. col rank = row rank = rank

# of lin ind cols = # of lin ind rows

PROOF:

WTS ① col rank $\le$ row rank
② row rank $\le$ col rank

} shows equality

define col rank $= k$, row rank $= r$

② A has col rank $k \Rightarrow \boxed{A = CV}$

$C \in \mathbb{R}^{m \times k}$ $\quad A = C[V_1 \cdots V_n]$ $\quad V \in \mathbb{R}^{k \times n}$

cols are lin ind.

lin ind

coeffs of the cols of A relative to C

$R(C) = R(A)$

$$A = [A_1 \cdots A_n] = [CV_1 \cdots CV_n]$$

cols of C = basis for R(A)

cols of V = coeffs

$A_1 = CV_1 \quad \cdots \quad A_n = CV_n$

trick. $\underline{A} = \underline{\bar{C}\,\bar{V}}$

change perspective from cols to rows...

$$C = \begin{bmatrix} - \bar{c}_1^T - \\ \vdots \\ - \bar{c}_m^T - \end{bmatrix} \qquad V = \begin{bmatrix} - \bar{v}_1^T - \\ \vdots \\ - \bar{v}_k^T - \end{bmatrix}$$

rows of V = span $R(A^T)$

rows of C = coeffs

coeffs of the rows of A relative to the rows of V

every row of A is a linear comb of the rows of V

k rows

$$A = \begin{bmatrix} - \bar{a}_1^T - \\ \vdots \\ - \bar{a}_m^T - \end{bmatrix} = \begin{bmatrix} \bar{c}_1^T V \\ \vdots \\ \bar{c}_m^T V \end{bmatrix}$$

$\bar{a}_1^T = \bar{c}_1^T V \qquad \bar{a}_m^T = \bar{c}_m^T V$

span of rows of A must have dim less than or equal to k

row rank $\leq$ col rank

$r \leq k$

1 similarly

$A = WR$ where rows of R are lin ind.

and $R(R^T) = R(A^T)$

start w rows

↓

cols

$k \leq r$

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Systems of Linear Eqns

$A \in \mathbb{R}^{m \times n}$

full col rank

$y = Ax$

"A is tall"

$m > n$

probably

no solutions

Condition for soln to exist.

$y \in R(A)$ ← span of cols

(only n lin ind rows)

↑

get an approx. solution

LEAST squares

full rank

$y = Ax$

A is square and invertible

$x = A^{-1}y$

unique solution

full row rank

$y = Ax$

"A is fat"

$m < n$

↓

infinite solutions

subspace of solutions

$\Rightarrow$ A has a nonzero nullspace

so if

$y = Ax_0$

and $x' \in N(A)$

select between solns. →

$x_0 + x'$ is a solution

$y = A(x_0 + x')$

$= Ax_0 + \cancel{Ax'}^{\to 0}$

minimum norm solution

if $R(N) = N(A)$

$x = \underset{\text{specific soln}}{x_0} + \underset{\text{in nullspace}}{\underline{Nz}}$

DATA ANALYSIS

y, A data points

fitting parameters x

CONSTRAINTS IN OPTIMIZATION PROBS. OR CONTROL PROBLEMS

→ LIMITS ON DEG OF FREEDOM

A physical eqns

A tall:

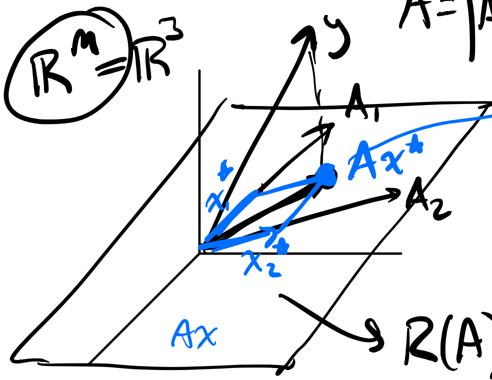
$A \in \mathbb{R}^{m \times n}$

$y = Ax$

$x \in \mathbb{R}^n$

Ex. $A \in \mathbb{R}^{3 \times 2}$ $\quad A = [A_1 \, A_2]$

$\mathbb{R}^m = \mathbb{R}^3$



→ $R(A)$

$$Ax^* = A(A^TA)^{-1}A^Ty$$
$$= \mathrm{proj}_A y$$

$$|y - Ax|^2 = \sum_i (y_i - [Ax]_i)^2$$

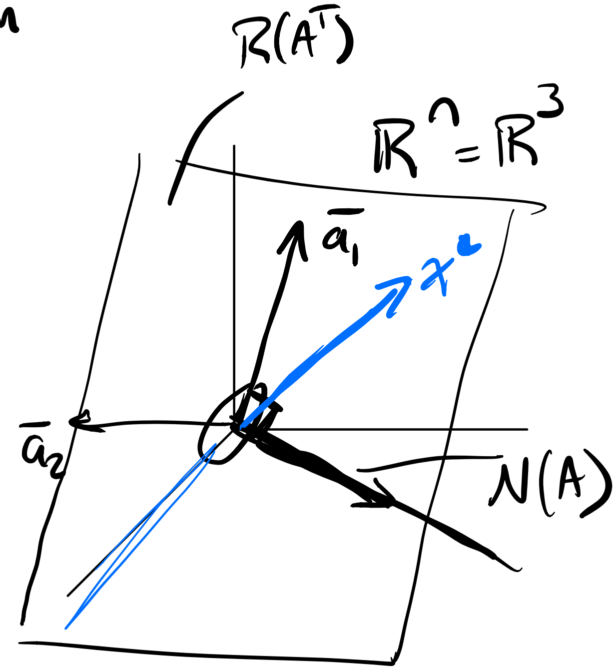
$$\min_x |y - Ax|^2 = (y - Ax)^T(y - Ax) = J$$
$$y^Ty - 2y^TAx + x^TA^TAx$$

LEAST SQUARES SOLN

$$\frac{\partial J}{\partial x} = -2y^TA + 2x^TA^TA = 0$$

$$x^T(A^TA) = y^TA \implies \boxed{x^* = (A^TA)^{-1}A^Ty}$$

A Sol. $A \in \mathbb{R}^{m \times n}$



Ex. $A \in \mathbb{R}^{2 \times 3}$

$$A = \begin{bmatrix} — \bar{a}_1^T — \\ — \bar{a}_2^T — \end{bmatrix}$$

$y = Ax$

if x_0 $\quad y = Ax_0$

$y = A(x_0 + x')$

$x' \in N(A)$

$\min_x |x|^2 = J$ $\leftarrow$ minimum norm soln to $y = Ax$

s.t. $y = Ax$

Lagrange Multipliers.

$L(x, \lambda) = J + \lambda^T (y - Ax) = x^T x + \lambda^T (y - Ax)$

$\frac{\partial L}{\partial x} = 0 \quad (2x^T - \lambda^T A = 0) A^T \Rightarrow 2x^T A^T = \lambda^T \underset{\sim}{AA^T}$

$\lambda^T = 2x^T A^T (AA^T)^{-1}$

$\frac{\partial L}{\partial \lambda} = 0 \quad y - Ax = 0$

$$\lambda = 2(AA^T)^{-1}Ax = 2(AA^T)^{-1}y$$

applying $y = Ax$

$$2x = A^T\lambda = 2A^T(AA^T)^{-1}y.$$

$$\boxed{x^* = A^T(AA^T)^{-1}y} \Leftarrow$$

minimum norm solution to $y = Ax$

if $x' \in N(A)$

$$(x')^T x^* = 0 \qquad \underbrace{(x')^T A^T}_{0}(AA^T)^{-1}y$$