Univ. of Washington

# Lecture : Vector Fields and Linear ODEs

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## 0.1 Ordinary Differential Equations (ODEs) and Vector Fields

We model time evolution of a system with *ordinary differential equations* (ODEs). Let $x \in \mathbb{R}^n$ be the vector valued *state* of some system that changes with time. A differential equation is written

$$\tfrac{\partial x(t)}{\partial t} = \dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function of $x$ and $x_0 \in \mathbb{R}^n$ is some initial state or *initial condition* of the system. If the system has a control input $u(t) \in \mathbb{R}^m$, some signal that we get to choose over time to modify the system dynamics, we can write

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0, \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is now a vector-valued function of $x$ and $u$. We often refer to differential equations of this form as *vector fields* cause we think of $f(x)$ as defining an evolution direction at each different point $x$ in the state space. We can represent this graphically as a "field" of arrows. $x(t)$ evolves forward in time along these arrows from the initial state $x_0$ to form a *state trajectory*.

The best understood differential equations are *linear differential equations* that have the form

$$\dot{x} = Ax, \qquad x(0) = x_0 \tag{3}$$

Linear differential equations are nice in that we can compute one *state transition matrix* or state evolution matrix that allows us $x(t)$ for *any initial condition*. In the simple case of (**??**) this transition matrix is given by $e^{At}$ and we have that

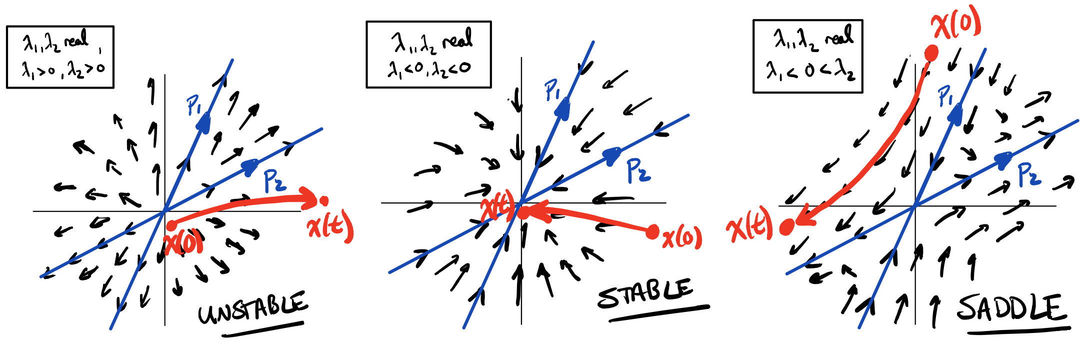
$$x(t) = e^{At}x(0) \tag{4}$$

If $A$ is diagonalizable, ie. $A = PDP^{-1}$ for eigenbasis $P$ then by the spectral mapping theorem, we have that $e^{At} = Pe^{Dt}P^{-1}$. If we represent $x$ in the eigenvector coordinates, ie. $x(t) = Pz(t)$ or $z(t) = P^{-1}x(t)$ this equation becomes

$$z(t) = e^{Dt}z(0) \qquad \to \qquad \begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} z_1(t) \\ \vdots \\ e^{\lambda_n t} z_n(t) \end{bmatrix} \tag{5}$$

ie. we can examine the evolution of each eigenvector separately. Suppose $\lambda = a + bi$. The exponential $e^{\lambda t} = e^{(a+bi)t} = e^{at}e^{bti}$. Thus $a$ controls the decay (or explosion) rate of the signal and $b$ (referred to as the *frequency*) determines the oscillation rate of the signal.

First, suppose all the eigenvalues of a real matrix $A$ are real. The possible vector fields and state trajectories (for $A \in \mathbb{R}^{2\times 2}$ are illustrated below.




$$\dot{x} = A x \qquad A = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1}$$

$$x(t) = e^{At} x(0) \qquad e^{At} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1}$$