Univ. of Washington

Lecture : Rotational Motion

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Lecturer: Dan Calderone

1 Rotational Motion

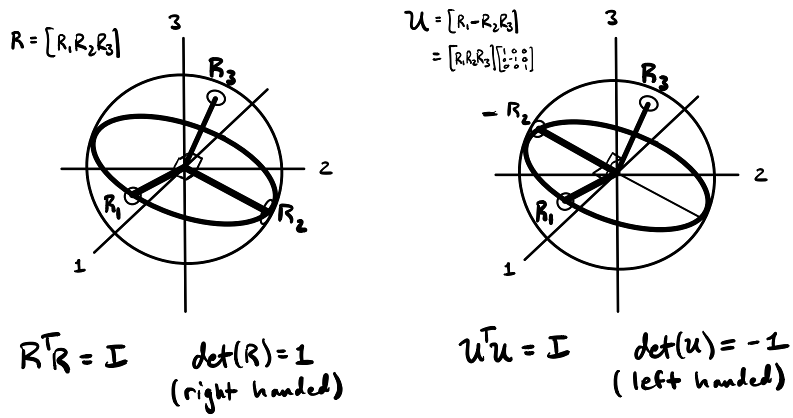
1.1 Rotation Matrices

A *rotation* matrix is a real valued matrix whose determinant is 1, and whose columns are *orthonormal*, ie. whose columns are orthogonal and have length 1. Note that these conditions can be succinctly written as

$$\text{det}(R) = 1, \qquad R^T R = I \tag{1}$$

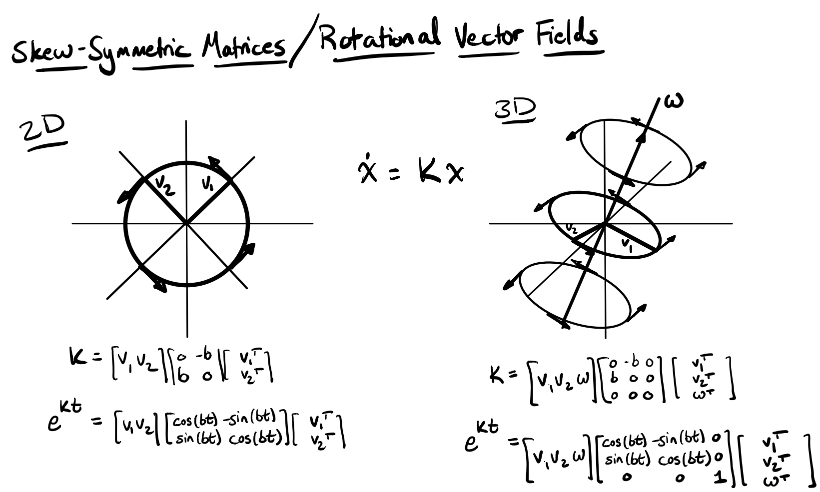
This second condition is that the left-inverse of R is its transpose, ie. $R^T = R^{-1}$. Note that this also means that $RR^{-1} = RR^T = I$, ie. that the rows of R are orthonormal as well. The generalization of rotation matrices to complex matrices are called *unitary* matrices, ie. $U \in \mathbb{C}^{n\times n}$ that satisfy det$(U) = 1$ and $U^*U = I$. The columns of a rotation matrix often used to represent an orthonormal coordinate system for $\mathbb{R}^n$. An orthonormal coordinate transformation is a type of *isometry*, a coordinate transformation that does not change metric of the space. One can see this by considering the coordinate transform $x = Rz$. All metric properties (distances and angles) are computed using inner products. Note that $z^T z = x^T R^T R x = x^T x$. As a result inner products in the x or z coordinates are the same so no metric properties change by transforming between these coordinate systems. This is consistent with our intuition about rotation matrices.

Relaxing the determinant restriction to $\det(R) = \pm 1$ allows the set of matrices to include rotations and reflections. This allows for left-handed as well as right-handed coordinate systems. A matrix R such that $R^T R = I, \ \det R = -1$ represents a left-handed coordinate system instead of a right-handed one. In order to transform from the standard basis coordinates to a left-handed coordinate system at least one reflection is required. Volumes transformed by orthogonal matrices such that $\det R = -1$ are flipped inside out.



1.2 Skew-Symmetric Matrices

A matrix $K \in \mathbb{R}^{n \times n}$ is *skew-symmetric* if $K = -K^T$. Skew-symmetric matrices have purely imaginary eigenvalues. Best thought of in the context of the equation $\dot{x} = Kx$. Note that you can prove explicitly that if $K = -K^T$ then $x^T K x = 0$, ie. Kx is perpendicular to x. Vector field rotational, ie. e^{Kt} is a rotation matrix. Eigenvalues come in complex conjugate pairs, so if you have an odd dimensional matrix then there is always one such that $\lambda = 0$. It follows that $e^{\lambda t} = e^{0t} = 1$ and the eigenvector associated with the 0 eigenvalue is the axis of rotation.



1.3 Matrix Lie Groups

Matrices can be thought of both as *vectors* or as *operators*. Sets of matrices are sometimes endowed with a local coordinate structure. These sets of matrices can be thought of as a *manifold* allowing one to talk about surface or space like properties of this set of matrices along with curvature and other geometric notions. Matrices as operators often represent transformations that encode symmetries, such as permutations, reflections, rotations, etc. In this context we can often talk about sets of matrices as mathematical *groups*. Group theory is interested in composing operations or transformations and seeing whether or not the composite operation has the same properties as the original two. In some case in the contexts of matrices, a set of matrices is both a manifold and a group. Perhaps the best example of these sets of matrices is the set of rotation matrices. Rotation matrices have a continuous manifold structure but also a clear set of symmetries that are preserved when two rotations are multiplied together. These matrix groups with a manifold structure are called *Lie groups*. Some well known examples are

- $GL(n)$: General linear group of dimension n, $n \times n$ matrices with $\det(A) = 1$ for $A \in GL(n)$.

  $\Rightarrow$) Lie algebra: $GL(n)$.

- $SO(n)$: Special orthogonal group of matrices $R \in \mathbb{R}^{n \times n}$ such that $R^T R = I$ and $\det R = 1$. This is the group of rotation matrices.

  $\Rightarrow$) Lie algebra: skew symmetric matrices.

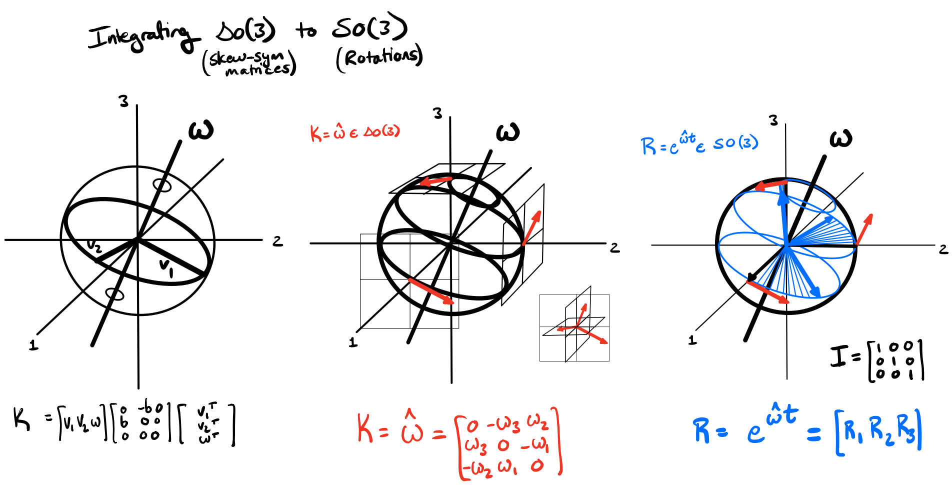
- $U(n)$: Unitary group of matrices $U \in \mathbb{C}^{n \times n}$ such that $U^T U = I$.

  $\Rightarrow$) Lie algebra: skew Hermitian matrices.

Rotation matrices are an excellent example of a Lie group. If $R_1, R_2 \in SO(n)$ then clearly $R_1 R_2 \in SO(n)$. In addition there exists a way to perturb a rotation matrix so that the new matrix is still a rotation, ie. there is some locally “flat” (Euclidean) structure to the space of matrices. This allows us to think of objects that lie in the tangent space to this manifold as infinitesimal rotations. The tangent space to the Lie group at the identity defines the *Lie algebra* of the Lie group. Integrating over an element in the Lie algebra builds up a particular element in the group. The Lie algebra for $SO(n)$ is the space of skew-symmetric matrices and the equation

$$R = e^{Kt}, \qquad \text{for} \quad K = -K^T$$

is an example of this integration. Geometrically, this concept can be visualizad as follows.



1.4 Complex Eigenvalues and Eigenvectors

Along with representing stretching, complex eigenvalues can represent rotation of vectors as well. If a real matrix has complex eigenvalues then they come in complex conjugate pairs. The eigenvectors come in conjugate pairs as well. We detail the exact mechanics of this below. Consider a matrix A with the first two eigenvalues and left and right eigenvectors given as

$$\lambda_1 = a + bi, \qquad r_1 = \tfrac{1}{\sqrt{2}}\big(u + vi\big), \qquad \ell_1 = \tfrac{1}{\sqrt{2}}\big(w + yi\big) \tag{2}$$

$$\lambda_2 = a - bi, \qquad r_2 = \tfrac{1}{\sqrt{2}}\big(u - vi\big), \qquad \ell_2 = \tfrac{1}{\sqrt{2}}\big(w - yi\big) \tag{3}$$

with real vectors u, v, w, y. The diagonal form of the matrix is given by

$$A = \begin{bmatrix} | & | & \\ r_1 & r_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & \ell_1^* & - \\ - & \ell_2^* & - \\ & \vdots & \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} | & | & \\ \tfrac{1}{\sqrt{2}}\big(u + vi\big) & \tfrac{1}{\sqrt{2}}\big(u - vi\big) & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} a + bi & 0 & \cdots & 0 \\ 0 & a - bi & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & \tfrac{1}{\sqrt{2}}\big(w - yi\big)^T & - \\ - & \tfrac{1}{\sqrt{2}}\big(w + yi\big)^T & - \\ & \vdots & \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} | & | & \\ u & v & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} a & b & \cdots & 0 \\ -b & a & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & w^T & - \\ - & y^T & - \\ & \vdots & \end{bmatrix} \tag{6}$$

This last equality is not obvious so we detail it below, but first note the form of the diagonal block, with the real parts of the eigenvalues on the diagonal and the imaginary parts on the off diagonal. Define the 2×2 complex matrix

$$U = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \tag{7}$$

Note that U is a unitary matrix, ie. $UU^* = I$. The first two dyads of the diagonalization are given by

$$\sum_{i=1,2} \lambda_i r_i \ell_i^* = \begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}(u - vi) & \frac{1}{\sqrt{2}}(u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(w+yi)^T & - \\ - & \frac{1}{\sqrt{2}}(w-yi)^T & - \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{U^*} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}}}_{U} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{U^*} \begin{bmatrix} -w^T- \\ -y^T- \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} -w^T- \\ -y^T- \end{bmatrix} \tag{10}$$

Note that both

$$\begin{bmatrix} -w^T- \\ -y^T- \end{bmatrix} \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \frac{1}{2} \begin{bmatrix} -(w+yi)^T- \\ -(w-yi)^T- \end{bmatrix} \begin{bmatrix} | & | \\ u - vi & u + vi \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} -w^T- \\ -y^T- \end{bmatrix} \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

We could write the eigenvalues as well in polar form $\lambda_1 = a + bi = \gamma e^{i\theta}$ and $\lambda_2 = a - bi = \gamma e^{-i\theta}$ where $\gamma = |\lambda_1| = |\lambda_2| = \sqrt{a^2 + b^2} \geq 0$. If we consider the diagonal block associated with the complex eigenvectors we have that

$$\begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \tag{13}$$

$$\gamma \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \gamma \underbrace{\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}}_{R} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \tag{14}$$

remembering that $e^{i\theta} = c\theta + s\theta i$. Note that R is rotation matrix. From this form, we can see that complex eigenvalues of a real matrix correspond to rotations and stretching. The stretching parameter is given by γ, the rotation is given by the matrix R and the corresponding angle θ, and the plane of rotation is the subspace spanned by the vectors u and v. The relationship of the vectors u and v (particularly the value of the inner product $u^T v$) determines the shape of the rotation. If u and v are orthogonal ($u^T v = 0$), then the rotation is circular. If not, then the rotation has an ellipsoidal shape.

Note that right and left eigenvector pairs are not unique but can be scaled by either a real or complex value. This is because the equation $\lambda v = Av$ does not specify the length of v. In diagonal form, since diagonal matrices commute, we can write (shown in here in the 2×2 case, but easily extended)

$$\begin{bmatrix} | & | \\ r_1 & r_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & \ell_1^* & - \\ - & \ell_n^* & - \end{bmatrix} = \begin{bmatrix} | & | \\ r_1 & r_2 \\ | & | \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{bmatrix} \begin{bmatrix} - & \ell_1^* & - \\ - & \ell_2^* & - \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} | & | \\ \alpha_1 r_1 & \alpha_2 r_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & \alpha_1^{-1}\ell_1^* & - \\ - & \alpha_2^{-1}\ell_2^* & - \end{bmatrix} \tag{16}$$

Thus we can scale the right eigenvectors by α_i and the left eigenvectors by α_i^{-1} and the diagonal form of the matrix stays the same. Note that in general α_i can be either real or complex. One interesting special case is the case of a real matrix with complex eigenvalues where $\alpha = e^{i\phi}$

$$\begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}(u - vi) & \frac{1}{\sqrt{2}}(u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \begin{bmatrix} \gamma e^{i\theta} & 0 \\ 0 & \gamma e^{-\theta i} \end{bmatrix} \begin{bmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(w + yi)^T & - \\ - & \frac{1}{\sqrt{2}}(w - yi)^T & - \end{bmatrix} =$$

$$= \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix} \gamma \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\phi & s\phi \\ -s\phi & c\phi \end{bmatrix} \begin{bmatrix} - & w^T & - \\ - & y^T & - \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} | & | \\ c\phi u + s\phi v & -s\phi u + c\phi v \\ | & | \end{bmatrix} \gamma \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} - & c\phi w^T + s\phi y^T & - \\ - & -s\phi w^T + c\phi y^T & - \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} | & | \\ u' & v' \\ | & | \end{bmatrix} \gamma \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} - & w'^T & - \\ - & y'^T & - \end{bmatrix} \tag{19}$$

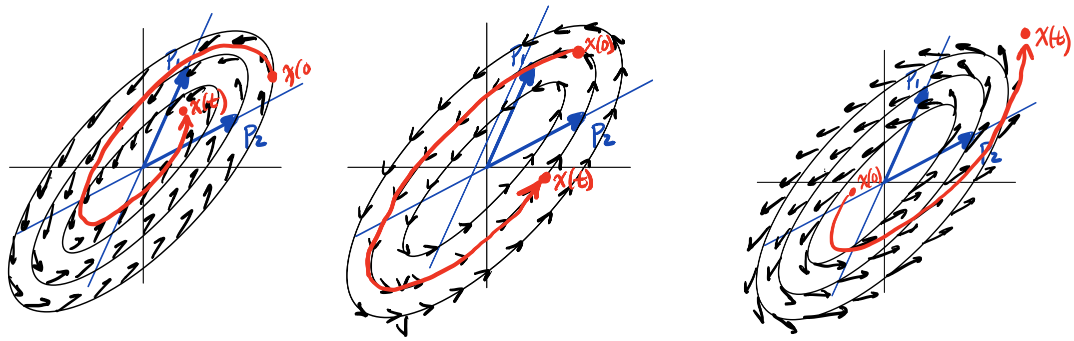
with $u' = c\phi u + s\phi v$, $v' = -s\phi u + c\phi v$, $w' = c\phi w + s\phi y$, $y' = -s\phi w + c\phi y$.

Note that from this analysis, we can see that the vectors u and v (and also w and y) are not unique but can be rotated by some phase ϕ. This can be derived either from the fact that diagonal matrices commute or from the fact that 2-D rotation matrices commute. Indeed, computing the quantity

$$\begin{bmatrix} | & | \\ u' & v' \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ c\phi u + s\phi v & -s\phi u + c\phi v \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} c\phi & s\phi \\ -s\phi & c\phi \end{bmatrix} \tag{20}$$

is equivalent to changing the basis vectors that span the subspace containing u and v. It is not obvious but a more complicated analysis (not shown) gives that u' and v' give the same ellipsoidal shape of rotation as u and v.

$\lambda_1 = a+bi$, $\lambda_2 = a-bi$, $a<0$	$\lambda_1 = bi$, $\lambda_2 = -bi$, $a=0$	$\lambda_1 = a+bi$, $\lambda_2 = a-bi$, $a>0$



$$\dot{x} = Ax \qquad A = \frac{1}{\sqrt{2}}\begin{bmatrix} p_1+p_2 i & p_1-p_2 i \end{bmatrix}\begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}\begin{bmatrix} p_1+p_2 i & p_1-p_2 i \end{bmatrix}^{-1}\sqrt{2}$$

$$= \begin{bmatrix} p_1 & p_2 \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1}$$

$$x(t) = e^{At}x(0) \qquad e^{At} = \frac{1}{\sqrt{2}}\begin{bmatrix} p_1-p_2 i & p_1+p_2 i \end{bmatrix}\begin{bmatrix} e^{at}e^{bti} & 0 \\ 0 & e^{at}e^{bti} \end{bmatrix}\begin{bmatrix} p_1-p_2 i & p_1+p_2 i \end{bmatrix}^{-1}\sqrt{2}$$

$$= \underbrace{\begin{bmatrix} p_1 & p_2 \end{bmatrix}}_{\text{shape of ellipses}}\underbrace{e^{at}}_{\text{scaling by } e^{at}}\underbrace{\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}}_{\text{rotation by } bt}\begin{bmatrix} p_1 & p_2 \end{bmatrix}^{-1}$$