AE 510 - Linear System Theory - Winter 2020

Homework 1

Due Date: Thursday, Jan 16^{th}, 2020 at 11:59 pm

1. **Inner Products**

   (a) **(PTS: 0-2)** Prove $y^T x = |x||y|\cos\theta$ using the definition of the 2-norm and the law of cosines.

   (b) **(PTS: 0-2)** Prove the parallelogram law:

   $$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$

2. **Projections**

   (a) **(PTS: 0-2)** Compute the projection of $x = [1, 2, 3]^T$ onto $y = [1, 1, -2]^T$.

   (b) **(PTS: 0-2)** Compute the projection of $x = [1, 2, 3]^T$ onto the range of

   $$Y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. **Block Matrix Computations**

   Multiply the following block matrices together. In each case give the required dimensions of the sub-blocks of B. If the dimensions are not determined by the shapes of A, then pick a dimension that works.

   (a) **(PTS: 0-2)**

   $$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1K} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NK} \end{bmatrix}, \qquad AB = ? \tag{1}$$

   where $A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{1N} \in \mathbb{R}^{m_1 \times n_N}$, $A_{M1} \in \mathbb{R}^{m_M \times n_1}$, and $A_{MN} \in \mathbb{R}^{m_M \times n_N}$.

   (b) **(PTS: 0-2)**

   $$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad AB = ? \tag{2}$$

   where $A_1 \in \mathbb{R}^{1 \times n}$ and $A_m \in \mathbb{R}^{1 \times n}$.

(c) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad AB = ? \tag{3}$$

where $A_1 \in \mathbb{R}^{m\times 1}$ and $A_n \in \mathbb{R}^{m\times 1}$.

(d) **(PTS: 0-2)**

$$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad D \in \mathbb{R}^{n\times n}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad ADB = ? \tag{4}$$

where $A_1 \in \mathbb{R}^{1\times n}$, $A_m \in \mathbb{R}^{1\times n}$.

(e) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad ADB = ? \tag{5}$$

where $A_1 \in \mathbb{R}^{m\times 1}$, $A_n \in \mathbb{R}^{m\times 1}$, $d_{ij} \in \mathbb{R}$.

(f) **(PTS: 0-2)**

$$A \in \mathbb{R}^{m\times n}, \quad \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}, \qquad AB = ? \tag{6}$$

(g) **(PTS: 0-2)**

$$A = \begin{bmatrix} -A_1- \\ \vdots \\ -A_m- \end{bmatrix}, \quad B, \qquad AB = ? \tag{7}$$

where $A_1, A_m \in \mathbb{R}^{1\times n}$.

4. **Norms**

   (PTS: 0-2) Compute the p-norm of the vector $x = \begin{bmatrix} -1 & 2 & 3 & -2 \end{bmatrix}^T$ for $p = 1, 2, 10, 100, \infty$.

5. **Vector Derivatives**

   Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. Compute $\frac{\partial f}{\partial x}$ for the following functions:

   (a) **(PTS: 0-2)**

$$f(x) = x_1^4 + 3x_1x_2^2 + e^{x_2} + \frac{1}{x_1x_2}$$

(b) **(PTS: 0-2)**

$$f(x) = \begin{bmatrix} \beta x_1 + \alpha x_2 \\ \beta(x_1 + x_2) \\ \alpha^2 x_1 + \beta x_2 \\ \beta x_1 + \frac{1}{\alpha} x_2 \end{bmatrix}$$

for $\alpha, \beta \in \mathbb{R}$

(c) **(PTS: 0-2)**

$$f(x) = \begin{bmatrix} e^{x^T Q x} \\ (x^T Q x)^{-1} \end{bmatrix}$$

for some $Q = Q^T \in \mathbb{R}^{2\times 2}$.

6. **Coordinates**

Let y be the coordinates of a vector with respect to the standard basis in $\mathbb{R}^2$. In each case below consider a different basis for $\mathbb{R}^2$ given by the columns of the matrix T. Compute the coordinates of the vector y with respect to the new basis 1) by graphically drawing the columns of T and y as vectors and 2) by inverting the matrix T, ie. by solving $y = Tx$.

(a) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

(c) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

(d) **(PTS: 0-2)** Graphical. **(PTS: 0-2)** Inverting T.

$$y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

7. **Elementary Matrices and Matrix Inverses**

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(a) **(PTS: 0-2)** Compute a sequence of elementary matrices that could be used to row-reduce A to the identity.

(b) **(PTS: 0-2)** Use this sequence of elementary matrices to compute A^{-1}.

8. **Symmetric and Skew Symmetric Matrices**

(a) **(PTS: 0-2)** Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. If A is a symmetric matrix, show that

$$\mathbf{a}^T A \mathbf{b} = \mathbf{b}^T A \mathbf{a}.$$

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(b) **(PTS: 0-2)** Show that if $C \in \mathbb{R}^{n \times n}$ is skew-symmetric, i.e. $C^T = -C$, then $\mathbf{x}^T C \mathbf{x} = 0$ for all $x \in \mathbb{R}^n$.

(c) **(PTS: 0-2)** Consider the quadratic expression $\mathbf{x}^T A \mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Show that for all A, there exists a matrix $B \in \mathbb{R}^{n \times n}$ with $B^T = B$ such that

$$\mathbf{x}^T B \mathbf{x} = \mathbf{x}^T A \mathbf{x}.$$

(Hint: You can use the result from part b.)