AE 510 - Linear Systems Theory - Winter 2020

Homework 2

Due Date: Thursday, Jan 23^{rd}, 2020 at 11:59pm

1. **Rotation Matrices**

   (a) **(PTS: 0-2)** Consider $R \in \mathbb{R}^{n\times n}$. Show that if R is a rotation matrix, then its inverse is also a rotation matrix.

   (b) **(PTS: 0-2)** Consider $R_1, R_2 \in \mathbb{R}^{n\times n}$ and $R = R_1R_2$. Prove that if R_1 and R_2 are rotation matrices, then R is also a rotation matrix.

   (c) **(PTS: 0-2)** *Three parameters* are sufficient to describe an arbitrary rotation matrix in 3D. Consider the rotation matrix $R \in \mathbb{R}^{3\times 3}$ defined in (1).

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{1}$$

   Assuming that $r_{13}, r_{23} \neq 0$ and $\vartheta \in (0, \pi)$, obtain the parameters defined in Fig.1, φ, ϑ and ψ, in terms of the elements of R.
   (Hint: You may find the function atan2 useful.)

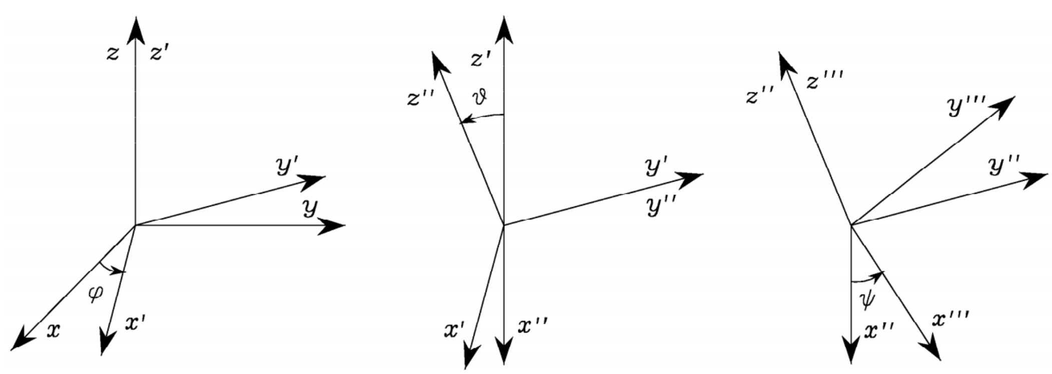

Figure 1: Rotation obtained as composition of three elementary rotations: rotation by the angle φ about z-axis, rotation by the angle ϑ about y'-axis and rotation by the angle ψ about z''-axis, respectively.

2. **Linear Transformations of Sets**

   Consider the unit-balls defined by the 1-norm, the 2-norm, and the $\infty - norm$

$$\mathcal{X}_1 = \left\{ x \;\middle|\; |x|_1 \leq 1 \right\}, \qquad \mathcal{X}_2 = \left\{ x \;\middle|\; |x|_2 \leq 1 \right\}, \qquad \mathcal{X}_\infty = \left\{ x \;\middle|\; |x|_\infty \leq 1 \right\}$$

   Draw the set of points Ax for $x \in \mathcal{X}_1$, $x \in \mathcal{X}_2$, and $x \in \mathcal{X}_\infty$ for

(PTS: 0-2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **(PTS: 0-2)** $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, **(PTS: 0-2)** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

3. **Matrix Representation Theorem**

   (PTS: 0-2) Consider a linear map $\mathcal{A} : R^3 \to \mathbf{P}_2$ where $\mathbf{P}_2$ is the space of quadratic polynomials, ie. polynomials of the form $\alpha_0 + \alpha_1 s + \alpha_2 s^2$. Consider the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ and the basis $\{1, s, s^2\}$ for $\mathbf{P}_2$. Suppose

$$\mathcal{A}(e_1) = 3 + 2s - s^2, \qquad \mathcal{A}(e_2) = -1 + s + s^2, \qquad \mathcal{A}(e_3) = 1 + s^2,$$

   Find a matrix representation $A \in R^{3\times3}$ of the map $\mathcal{A}$, such that $y = Ax$ where if $x \in R^3$ are the coordinates of a vector in R^3 with respect to the standard basis, then $y \in R^3$ are the coordinates of $\mathcal{A}(v)$ with respect to the basis $\{1, s, s^2\}$ of P_2.

4. **Similarity Transformations**

   Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   and the equation $y = Ax$ for $x, y \in R^2$. For each coordinate transformation $T \in R^{2\times2}$ shown below, compute the matrix A' such that $y' = A'x'$ when $x = Tx'$ and $y = Ty'$.

   (PTS: 0-2) $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, **(PTS: 0-2)** $T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$,

5. **Nullspace**

   (a) **Basis Derivation**

   Consider a fat matrix $A \in R^{m\times n}$ $(m < n)$ that is partitioned as $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ with $A_1 \in R^{m\times m}$. Show that the columns of $B \in R^{n\times n-m}$

$$B = \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix}$$

   form a basis for the nullspace of A, $\mathcal{N}(A)$ by performing the following two steps.

   i. **(PTS: 0-2)** Show that any vector $v \in \mathcal{N}(A)$ can be written as $v = Bw$ for some $w \in R^{n-m}$, ie. v is linear combination of the columns of B (the columns of B span the nullspace).

   ii. **(PTS: 0-2)** Show that the columns of B are linearly independent.

   (b) **Computation**

   For the following matrices explicitly compute a basis for their nullspaces BY HAND, ie. do not use computational software.

   i. **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

ii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix}$$

iii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{bmatrix}$$

6. **Matrix Rank**

   The column rank of a matrix is the number of linearly independent columns. The row rank of a matrix is the number of linearly independent row.

   (a) **(PTS: 0-2)** Show that the row rank is less than or equal to the column rank.

   (b) **(PTS: 0-2)** Show that the col rank is less than or equal to the row rank.