AE 510 - Linear Systems Theory - Winter 2020

Homework 3

Due Date: Thursday, Jan 30^{th}, 2020 at 11:59pm

1. **Traces and Determinants**

   (a) **(PTS: 0-2)** Consider $A, B \in \mathbb{R}^{n\times n}$. Show that $\text{tr}\,(AB) = \text{tr}\,(BA)$.

   (b) **(PTS: 0-2)** Consider $A, B \in \mathbb{R}^{n\times n}$. Show that there do not exist matrices $A, B \in \mathbb{R}^{n\times n}$ such that $AB - BA = I$ where I is the identity matrix.

   (c) **(PTS: 0-2)**: The determinant of a diagonal matrix is the product of the diagonal elements. Draw (or describe) a picture illustrating this fact based on the determinant being the signed volume of the transformed unit cube.

   Assume that $A \in R^{n\times n}$ is diagonalizable and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. Use the properties of traces and determinants and the fact from part (a) to show that

   (d) **(PTS: 0-2)**: $\quad \text{Tr}(A) = \sum_i \lambda_i$

   (e) **(PTS: 0-2)**: $\quad \det(A) = \prod_i \lambda_i$

2. **Range and Nullspace**

   Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent the range and nullspace of A (and similarly let $\mathcal{R}(A^T)$ and $\mathcal{N}(A^T)$ be the range and nullspace of A^T).

   (a) **(PTS: 0-2)** Suppose $y \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$. Show that $x \perp y$, ie. $x^T y = 0$.

   (b) **(PTS: 0-2)** Consider $A \in R^{5\times 10}$. Suppose A has only 3 linearly independent columns (the other 7 are linearly dependent on the first 3). What is the dimension of $\mathcal{R}(A)$? What is the dimension of $\mathcal{N}(A^T)$?

   (c) **(PTS: 0-2)** What is the dimension of $\mathcal{N}(A)$? What is the dimension of $\mathcal{R}(A^T)$?

3. **Least Squares and Minimum Norm Solutions**

   (a) **(PTS: 0-2)** Consider $A \in R^{m\times n}$ where $m > n$ (A is "tall") and A has full-column rank (the columns are linear independent). Show that the least squares solution $x = (A^\intercal A)^{-1} A^\intercal y$, minimizes $|y - Ax|^2$, ie. makes Ax as close as possible to y.

   (b) **(PTS: 0-2)** Consider $A \in R^{m\times n}$ where $m < n$ (A is "fat") and A has full-row rank (the rows are linear independent). Let $x = A^\intercal (AA^\intercal)^{-1} y$ and $z \in R^n$ be any vector such that $y = Az$. Show that $|x| \leq |z|$.

4. **Similarity Transforms and Diagonalization**

   Suppose $p_1, p_2 \in R^2$ are linearly independent right eigenvectors of $A \in R^{2\times 2}$ with eigenvalues $\lambda_1, \lambda_2 \in R$ such that $\lambda_1 \neq \lambda_2$. Suppose that

   $$p_1^T p_2 = 0, \qquad |p_1| = 1, \qquad |p_2| = 2$$

(a) **(PTS: 0-2)** Write an expression for a 2×2 matrix whose rows are the left-eigenvectors of A

(b) **(PTS: 0-2)** Write an expression for a similarity transform that transforms A into a diagonal matrix.

5. **Spectral Mapping Theorem**

Consider a diagonalizable matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a polynomial function $f : R^{n\times n} \to R^{n\times n}$.

(a) **(PTS: 0-2)** Show that the eigenvectors (left and right) of $f(A)$ are the same as the eigenvectors of A.

(b) **(PTS: 0-2)** Show that the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

6. **Similar Eigenvalues**

(a) **(PTS: 0-2)** Let $A \in \mathbb{R}^{n\times n}$ and let $T \in \mathbb{R}^{n\times n}$ be any non-singular matrix. Show that the eigenvalues of A are the same as those of $T^{-1}AT$.

(b) **(PTS: 0-2)** Let $A, B \in \mathbb{R}^{n\times n}$ be invertible matrices. Show that the eigenvalues of AB are the same as those of BA.