AE 510 - Linear Systems Theory - Winter 2020

Homework 5

Due Date: Thursday, Feb 13^{th}, 2020 at 11:59pm

1. **Continuous vs. Discrete Time Stability**

   (PTS: 0-2) Let $\rho(M)$ represent the set of eigenvalues or *spectrum* of M. Show that if $\text{Re}(\lambda) < 0$ for all $\lambda \in \rho(A)$, then $|\mu|_2 < 1$ for all $\mu \in \rho(e^{A\Delta t})$.

2. **Solutions to Linear Dynamical Systems**

   (a) **(PTS: 0-2)** If

   $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\ d\tau$$

   Show that

   $$\dot{x} = \tfrac{d}{dt}x(t) = Ax(t) + Bu(t)$$

   (Hint: Leibniz integral rule will be helpful.)

   (b) **(PTS: 0-2)** Consider the discrete time (time varying) update equation

   $$x[t+1] = A[t]x[t] + B[t]u[t]$$

   Write an expression for $x[t]$ in terms of the initial state $x[0]$, $u[0], ..., u[t-1]$, $A[0], \ldots, A[t-1]$, and $B[0], \ldots, B[t-1]$.

3. **Discretizing Continuous Time Systems**

   For the given continuous time evolution equation of the form

   $$\dot{x} = Ax + Bu,$$

   compute the matrices $\bar{A}$ and $\bar{B}$ for the corresponding discrete time update equation

   $$x^+ = \bar{A}x + \bar{B}u$$

   for a time step $\Delta t = 0.01$ seconds.

   (a) **(PTS: 0-2)**

   $$\dot{x} = Ax + Bu, \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   (b) **(PTS: 0-2)**

   $$\dot{x} = Ax + Bu, \qquad A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. **Controllability**

   Consider the dynamical system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

   where $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$ and A is diagonalizable with right and left eigenvectors the columns and rows of P and Q respectively

$$P = \begin{bmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{bmatrix}, \qquad Q = P^{-1} = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \tag{1}$$

   (a) **(PTS: 0-2)** Suppose there exists a left eigenvector of A, $q_i^T \in R^{1\times n}$ such that $q_i^T B = 0$. Show that the system is not controllable.

   (b) **(PTS: 0-2)** Now suppose $q_i^T B \neq 0$ for all i, but the first two eigenvalues are the same. Show that the system is not controllable.

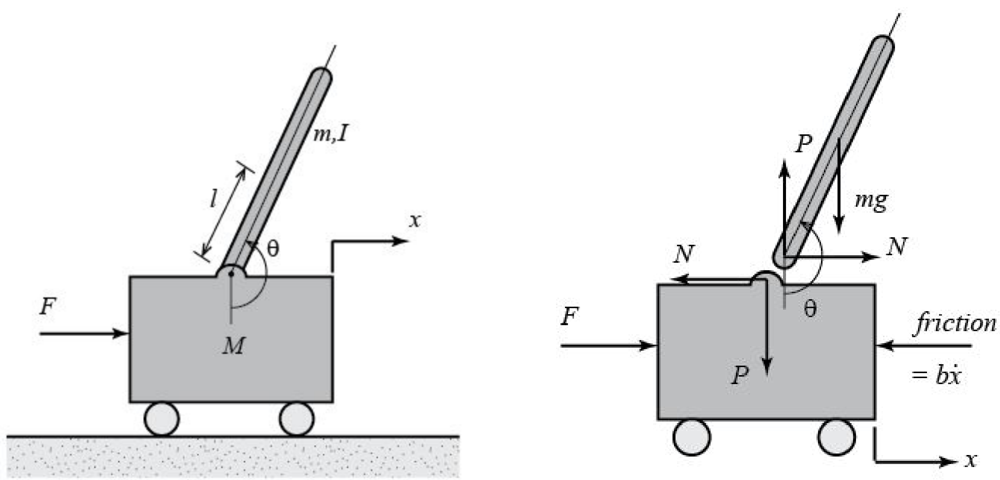
5. **Discrete Time Control of Inverted Pendulum**

   Consider the linearized model of an inverted pendulum shown below.

   (a) **(PTS: 0-2)** Compute the discrete time system for a time step size of $\Delta t = 0.01$ seconds.

   (b) **(PTS: 0-2)** Show that the discrete time system is controllable.

   (c) **(PTS: 0-2)** Compute the minimum norm open-loop control to drive the system to **0** from the initial condition shown below in 100 time steps (1 second).



   System Parameters

   (a) M = mass of cart 0.5 $[kg]$

   (b) m = mass of the pendulum 0.2 $[kg]$

   (c) b = coefficient of friction for cart 0.1 $[N/m/s]$

(d) l = length of pendulum center of mass 0.3 $[m]$

(e) I = mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

(f) F = force applied to the cart

(g) x = cart position coordinate

(h) θ = pendulum angle from vertical (down)

(i) $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$l\left(I + ml^2\right)\ddot{\phi} - mgl\phi = ml\ddot{x}$$
$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

State-space:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$x[0] = \begin{bmatrix} -3 \\ 2 \\ \frac{\pi}{8} \\ -\frac{\pi}{4} \end{bmatrix}$$