AE 510 - Linear Systems Theory - Winter 2020

Homework 6

Due Date: Friday, Feb 21^{th}, 2020 at 11:59pm

1. **Controllability/Observability: Coordinate Invariance**

   Consider a dynamical system

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du\end{aligned}$$

   The system is controllable and observable. Show that under the coordinate transformation $x = Tz$

   (a) **(PTS: 0-2)**. The system is still controllable in the z-coordinates.

   (b) **(PTS: 0-2)**. The system is still observable in the z-coordinates.

2. **Observability Test**

   Consider the dynamical system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

   where $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$, $D \in R^{1\times 1}$ and A is diagonalizable with right and left eigenvectors the columns and rows of P and Q respectively

$$P = \begin{bmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{bmatrix}, \qquad Q = P^{-1} = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \tag{1}$$

   (PTS: 0-2) Suppose there exists a right eigenvector of A, $p \in R^n$ such that $Cp = 0$. Show that the system is not observable.

3. **Feedback Control: Eigenvalue Placement**

   Consider the two systems shown below of the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

   with $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{o\times n}$, and $D \in R^{o\times m}$.

   For each system, follow the steps given to design a feedback gain matrix K that stabilizes the closed-loop system matrix $A + BK$

(a) **(PTS: 0-2)** Compute the characteristic polynomial of A.

$$\det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$$

Select (distinct) desired eigenvalues $\lambda_1, \ldots, \lambda_n$ for the closed loop system $A+BK$ so that the closed loop system will be stable. Compute the desired characteristic polynomial for $A+BK$ using the formula

$$\det(\lambda I - (A + BK)) = \Pi_i(\lambda - \lambda_i) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0$$

(b) **(PTS: 0-2)** If the system is controllable, compute a coordinate transformation $x = Tz$ such that the system in the z coordinates is in *controllable canonical form*

$$\dot{z} = \bar{A}z + \bar{B}u$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Use the fact that if T exists, then the controllability matrix in the two different coordinates are related by

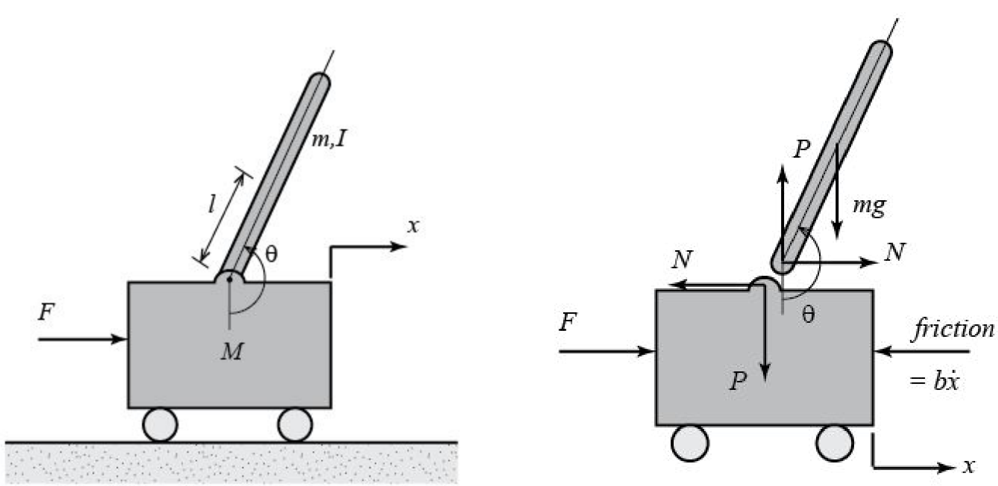
$$\begin{bmatrix} \bar{A}^{n-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix} = T^{-1} \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \tag{2}$$

(c) **(PTS: 0-2)** Compute the gain matrix $\bar{K}$ such that $\bar{A} + \bar{B}\bar{K}$ has the desired characterisitic polynomial. $\lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0$.

(d) **(PTS: 0-2)** Compute the feedback gain matrix K so that the closed loop system matrix $A + BK$ has the desired characteristic polynomial using $\bar{K}$ and T.

(e) **(PTS: 0-2)** Check that the closed-loop system matrix $A + BK$ is stable.

- **Inverted Pendulum**



System Parameters

(a) $M =$ mass of cart 0.5 $[kg]$

(b) $m =$ mass of the pendulum 0.2 $[kg]$

(c) $b =$ coefficient of friction for cart 0.1 $[N/m/s]$

(d) $l =$ length of pendulum center of mass 0.3 $[m]$

(e) $I =$ mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

(f) $F =$ force applied to the cart

(g) $x =$ cart position coordinate

(h) $\theta =$ pendulum angle from vertical (down)

(i) $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$l\big(I + ml^2\big)\,\ddot{\phi} - mgl\phi = ml\ddot{x}$$
$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

State-space:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

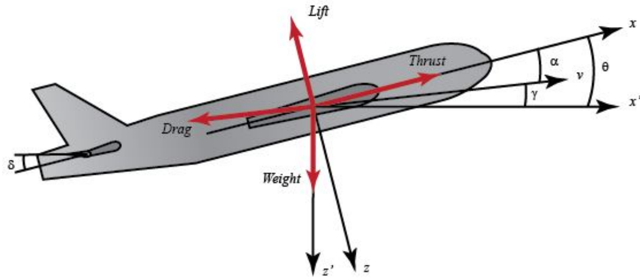
$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$x[0] = \begin{bmatrix} -3 \\ 2 \\ \frac{\pi}{8} \\ -\frac{\pi}{4} \end{bmatrix}$$

- **Aircraft Pitch**



System Parameters

$\alpha =$ angle of attack

$q =$ pitch rate

$\theta =$ pitch angle

$\delta =$ elevator deflection angle

$\mu = \frac{\rho S \bar{c}}{4m}$

$\rho =$ air density

$S =$ area of wing

$\bar{c} =$ mean chord length

$m =$ aircraft mass

$\Omega = \frac{2U}{\bar{c}}$

$U =$ equilibrium flight of speed

$C_T =$ Coefficient of Thrust

$C_D =$ Coefficient of Drag

$C_L =$ Coefficient of Lift

$C_W =$ Coefficient of Weight

$C_M =$ Coefficient of Pitch Moment

$\gamma =$ Flight path angle

$\sigma = \frac{1}{1+\mu C_L} =$ constant

$i_{yy} =$ normalized moment of inertia

$\eta = \mu \sigma C_M =$ constant

Equations of Motion:

$$\dot{\alpha} = \mu \Omega \sigma \left[-\left(C_L + C_D\right)\alpha + \frac{1}{\left(\mu - C_L\right)} q - \left(C_W \sin\gamma\right)\theta + C_L \right]$$

$$\dot{q} = \frac{\mu \Omega}{2 i_{yy}} \left[\left[C_M - \eta\left(C_L + C_D\right)\right]\alpha + \left[C_M + \sigma C_M \left(1 - \mu C_L\right)\right] q + \left(\eta C_W \sin\gamma\right)\delta \right]$$

$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$

$$x[0] = \begin{bmatrix} \frac{\pi}{16} \\ -\frac{\pi}{8} \\ \frac{\pi}{12} \end{bmatrix}$$

4. **Observability and Least Squares**

For the two systems in the previous problem, perform the following steps.

(a) **(PTS: 0-2)** Check whether or not the closed-loop system is observable.

(b) **(PTS: 0-2)** Simulate the trajectory forward using the feedback gain you computed in the previous problem from the initial condition given for 100 time steps with a time step size of $\Delta t = 0.01$ seconds. At each time t, compute $y[t]$.

(c) **(PTS: 0-2)** If the system is observable, use the output trajectory $y[0], \ldots, y[100]$ and the method of least squares to compute the initial condition $x(0)$.