AE 510 - Linear Systems Theory - Winter 2020

Homework 7

Due Date: Friday, Feb 28^{th}, 2020 at 11:59pm

1. **Laplace Transform**

   The Laplace transform of a function $f(t)$ is given by

   $$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\ dt$$

   (a) **Choose three** of the following Laplace Transforms to compute:
   - **(PTS: 0-2)** Delta function: $\mathcal{L}\left(\delta(t)\right) = ?$
   - **(PTS: 0-2)** Differentiation: $\mathcal{L}\left(\dot{f}(t)\right) = ?$
   - **(PTS: 0-2)** Integration: $\mathcal{L}\left(\int_0^t f(\tau)\ d\tau\right) = ?$
   - **(PTS: 0-2)** Frequency Shift: $\mathcal{L}\left(e^{at}f(t)\right) = ?$
   - **(PTS: 0-2)** Convolution: $\mathcal{L}\left(\int_0^t g(t-\tau)f(\tau)\ d\tau\right) = ?$

   (b) **(PTS: 0-2)** Find the Laplace transform of $y(t) = f(t) * h(t)$ where $*$ is the convolution operator.

   $$h(t) = e^{-t}, \quad t \geq 0$$
   $$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t \leq 2 \\ 0, & 2 < t \end{cases}$$

   (c) **(PTS: 0-2)** Obtain an approximation of $Y(s)$ from part (b) in the following form using the first-order Padé approximation.

   $$Y_{\text{approx}}(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}, \quad m \leq n$$

   Note: A Padé approximant is the "best" approximation of a function by a rational function of given order. The first-order Padé approximation of a time delay τ is

   $$e^{-s\tau} \approx \frac{1 - \frac{\tau}{2}s}{1 + \frac{\tau}{2}s}.$$

2. **Transfer Functions**

   Consider the continuous time linear system

   $$\dot{x} = Ax + Bu, \qquad x(0) = x_0$$
   $$y = Cx + Du$$

   with $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{o\times n}$, and $D \in R^{o\times m}$.

(a) **(PTS: 0-2)** Compute the Laplace transform of the output $y(t)$, ie. $\mathcal{L}(y(t)) = Y(s)$. Your solution should be in terms of the $A, B, C, D, x_0, U(s)$, where $U(s)$ is the Laplace transform of $u(t)$.

(b) **(PTS: 0-2)** Assuming A is diagonalizable, expand out your previous answer to show that

$$Y(s) = \sum_i \left(\frac{1}{s-\lambda_i} C p_i q_i^T x_0\right) + \left[\sum_i \left(\frac{1}{s-\lambda_i} C p_i q_i^T B\right) + D\right] U(s)$$

where p_i and q_i are the right and left eigenvectors for eigenvalue λ_i.

3. **Observer Design: Eigenvalue Placement**

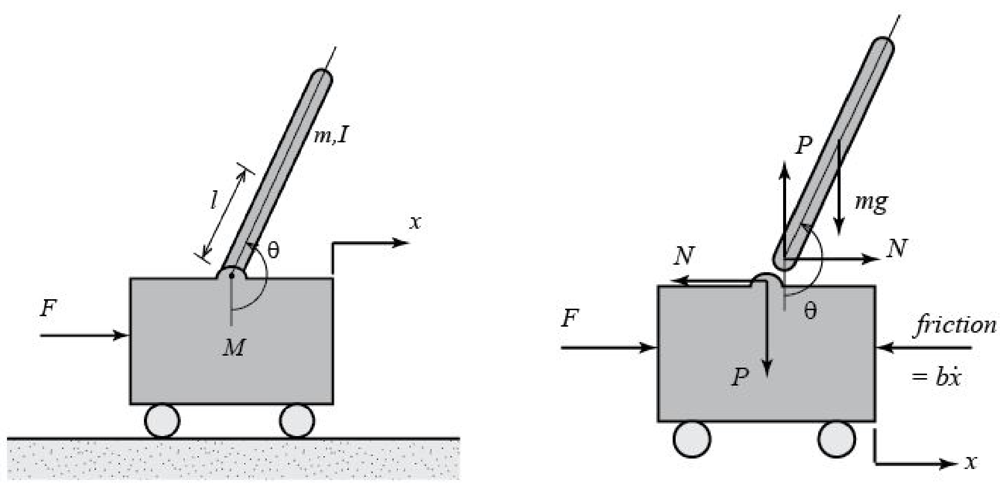
Consider the two systems shown below of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$, and $D \in R^{1\times 1}$.

For each system, design an observer gain $L \in R^{n\times 1}$ such that the matrix $A + LC$ is stable. Note: you should follow the steps of Problem 3 from Homework 6. The points for this problem will be assigned similarly.

- **Inverted Pendulum**



System Parameters

(a) $M =$ mass of cart 0.5 $[kg]$

(b) $m =$ mass of the pendulum 0.2 $[kg]$

(c) $b =$ coefficient of friction for cart 0.1 $[N/m/s]$

(d) $l =$ length of pendulum center of mass 0.3 $[m]$

(e) $I =$ mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

(f) $F =$ force applied to the cart

(g) $x =$ cart position coordinate

(h) $\theta =$ pendulum angle from vertical (down)

(i) $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$l\left(I + ml^2\right)\ddot{\phi} - mgl\phi = ml\ddot{x}$$
$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

State-space:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

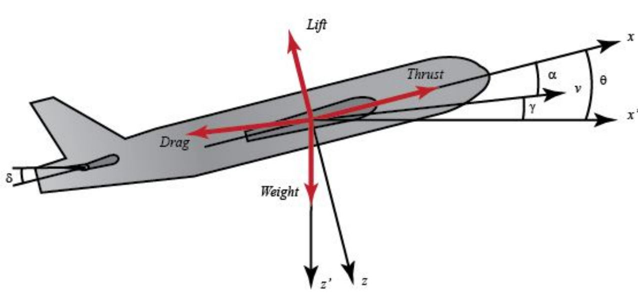
$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

$$x[0] = \begin{bmatrix} -3 \\ 2 \\ \frac{\pi}{8} \\ -\frac{\pi}{4} \end{bmatrix}$$

- **Aircraft Pitch**



System Parameters

α = angle of attack

θ = pitch angle

$\mu = \frac{\rho S \bar{c}}{4m}$

S = area of wing

m = aircraft mass

U = equilibrium flight of speed

C_D = Coefficient of Drag

C_W = Coefficient of Weight

γ = Flight path angle

i_{yy} = normalized moment of inertia

q = pitch rate

δ = elevator deflection angle

ρ = air density

$\bar{c}$ = mean chord length

$\Omega = \frac{2U}{\bar{c}}$

C_T = Coefficient of Thrust

C_L = Coefficient of Lift

C_M = Coefficient of Pitch Moment

$\sigma = \frac{1}{1+\mu C_L}$ = constant

$\eta = \mu \sigma C_M$ = constant

Equations of Motion:

$$\dot{\alpha} = \mu\Omega\sigma\left[-\left(C_L + C_D\right)\alpha + \frac{1}{(\mu - C_L)}q - \left(C_W \sin\gamma\right)\theta + C_L\right]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\left[\left[C_M - \eta\left(C_L + C_D\right)\right]\alpha + \left[C_M + \sigma C_M\left(1 - \mu C_L\right)\right]q + \left(\eta C_W \sin\gamma\right)\delta\right]$$

$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$

$$x[0] = \begin{bmatrix} \frac{\pi}{16} \\ -\frac{\pi}{8} \\ \frac{\pi}{12} \end{bmatrix}$$

4. **Observer Design: Separation principle**

For the two systems above

(a) **(PTS: 0-2)** Write down the estimator dynamics for a state estimate $\hat{x} \in R^n$. Note that these should include the observer gain L times the output estimate error.

(b) **(PTS: 0-2)** Write down the joint dynamics of the true state $x \in R^n$ and the estimator state $\hat{x} \in R^n$.

(c) **(PTS: 0-2)** Write down the coordinate transformation $T \in R^{2n \times 2n}$ such that

$$\begin{bmatrix} x \\ e \end{bmatrix} = T \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

where $e \in R^n$ is the error in the state estimate $e = \hat{x} - x$. Use this coordinate transformation to transform the dynamics from the previous part into dynamics for $\begin{bmatrix} x \\ e \end{bmatrix}$.

(d) **(PTS: 0-2)** Show that the stability of the joint dynamics depends on the stability of $A+BK$ and $A+LC$ separately.

(e) Simulate the joint the state-error system using the observer gain L computed above and the control input $u = K\hat{x} + r$ where K is the feedback gain computed in Homework 6 and r is each of the two following reference signals. Use the initial conditions $x(0)$ given with the dynamics and the initial state estimate $\hat{x}(0) = 0$.

- **(PTS: 0-2)** $r = 1$
- **(PTS: 0-2)** $r = \gamma \sin(\omega t)$ (Pick a γ and ω you find interesting.)

(f) **(PTS: 0-2)**($\times 2$) Plot the state, error, and control trajectories for each case.