AE 510 - Linear Systems Theory - Winter 2020

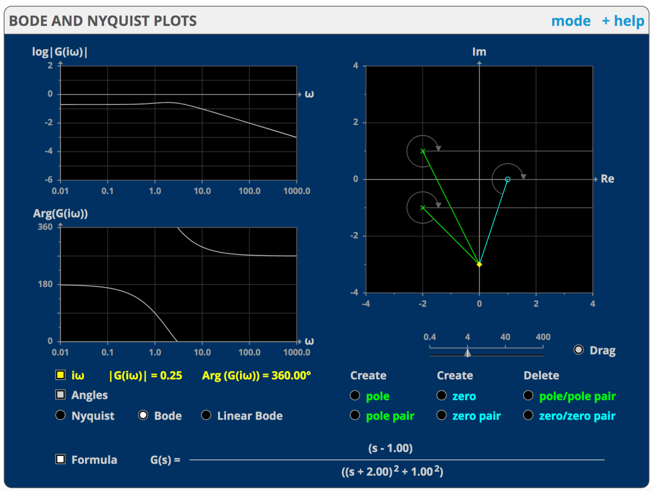
Homework 8

Due Date: Tuesday, Mar 10^{th}, 2020 at 11:59pm

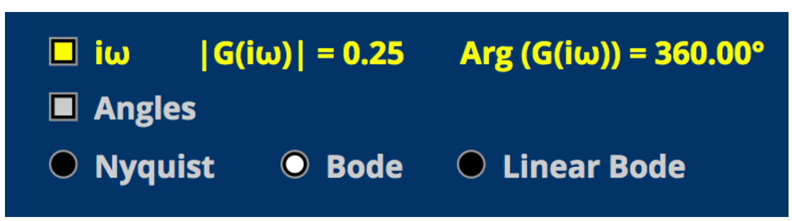
1. **Bode and Nyquist Plots**

   Use the tool in the link below to visualize the Bode plots for the transfer functions listed below.

   https://mathlets.org/mathlets/bode-and-nyquist-plots/



   Make sure to try out the $i\omega$ and **Angles** options.



   For each case, **take at least one screenshot and comment** on the behavior of the transfer function and how it is represented in the Bode plots.

   Note: The goal is to get intuition on how poles and zeros affect transfer function behavior. Make sure you spend as much or more time playing around with it as you do taking screenshots.

   (a) **(PTS: 0-2)** One real pole. Vary λ along the real axis. Specifically note what happens when $\lambda = 0$.

$$G(s) = \frac{1}{s - \lambda}$$

(b) **(PTS: 0-2)** A pair of complex poles. Vary the poles and specifically note what happens when they cross the $j\omega$-axis.

$$G(s) = \frac{1}{(s-\lambda_1)(s-\lambda_2)}$$

(c) **(PTS: 0-2)** One real pole and one real zero. Vary the zero z along the real axis. In particular note, the high frequency behavior of the transfer function.

$$G(s) = \frac{s-z}{s-\lambda}$$

(d) **(PTS: 0-2)** A pair of complex poles and a real zero. Vary the zero z along the real axis. In particular note what happens to the phase when the zero crosses the $j\omega$-axis.

$$G(s) = \frac{s-z}{(s-\lambda_1)(s-\lambda_2)}$$

(e) **(PTS: 0-2)** A pair of complex poles and a pair of complex zeros. Vary the zeros. Note what happens when the zeros cross the $j\omega$-axis.

$$G(s) = \frac{(s-z_1)(s-z_2)}{(s-\lambda_1)(s-\lambda_2)}$$

2. **Z-transforms**

(PTS: 0-2) Consider a function $f(t)$ and consider a new function $\bar{f}(t)$ obtained by sampling $f(t)$ every Δt seconds.

$$\bar{f}(t) = f(t)\delta(t-\Delta tk) = \begin{cases} f(\Delta tk) & ;\ \text{when } t = k\Delta t \text{ for } k = 0,1,2,\ldots \\ 0 & ;\ \text{otherwise} \end{cases}$$

Compute the Laplace transform of $\bar{f}(t)$ and show that you obtain the z-transform

$$\mathcal{L}(\bar{f}) = \sum_{k=0}^{\infty} f(\Delta tk) z^{-k}$$

where z is defined as $z = e^{s\Delta t}$.

3. **Circulant Matrices**

Consider the *shift* matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and the *circulant matrix* (for the vector $\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_4 c_5 \end{bmatrix}$).

$$C = \begin{bmatrix} c_0 & c_5 & c_4 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_5 & c_4 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_5 & c_4 & c_3 \\ c_3 & c_2 & c_1 & c_0 & c_5 & c_4 \\ c_4 & c_3 & c_2 & c_1 & c_0 & c_5 \\ c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \end{bmatrix}$$

(a) **(PTS: 0-2)** Check that the columns of V are right eigenvectors of P. **(You can just check 3 of them.)**

$$V = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{i2\pi(1\times1)}{6}} & e^{\frac{i2\pi(1\times2)}{6}} & e^{\frac{i2\pi(1\times3)}{6}} & e^{\frac{i2\pi(1\times4)}{6}} & e^{\frac{i2\pi(1\times5)}{6}} \\ 1 & e^{\frac{i2\pi(2\times1)}{6}} & e^{\frac{i2\pi(2\times2)}{6}} & e^{\frac{i2\pi(2\times3)}{6}} & e^{\frac{i2\pi(2\times4)}{6}} & e^{\frac{i2\pi(2\times5)}{6}} \\ 1 & e^{\frac{i2\pi(3\times1)}{6}} & e^{\frac{i2\pi(3\times2)}{6}} & e^{\frac{i2\pi(3\times3)}{6}} & e^{\frac{i2\pi(3\times4)}{6}} & e^{\frac{i2\pi(3\times5)}{6}} \\ 1 & e^{\frac{i2\pi(4\times1)}{6}} & e^{\frac{i2\pi(4\times2)}{6}} & e^{\frac{i2\pi(4\times3)}{6}} & e^{\frac{i2\pi(4\times4)}{6}} & e^{\frac{i2\pi(4\times5)}{6}} \\ 1 & e^{\frac{i2\pi(5\times1)}{6}} & e^{\frac{i2\pi(5\times2)}{6}} & e^{\frac{i2\pi(5\times3)}{6}} & e^{\frac{i2\pi(5\times4)}{6}} & e^{\frac{i2\pi(5\times5)}{6}} \end{bmatrix}$$

(b) **(PTS: 0-2)** What are the eigenvalues associated with each eigenvector?

(c) **(PTS: 0-2)** Which eigenvectors are conjugate pairs of each other?

(d) **(PTS: 0-2)** Show that the circulant matrix C can be written as

$$C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 + c_4 P^4 + c_5 P^5$$

Are the columns of V eigenvectors of C?

(e) **(PTS: 0-2)** Show that the columns of V are orthogonal to each other, ie. $V_i^* V_j = 0$ for $i \neq j$ where V_i and V_j are columns of V. What does this say about V^{-1}?

(f) **(PTS: 0-2)** Use the spectral mapping theorem to compute the eigenvalues of C.

(g) **(PTS: 0-2)** Write out a diagonalization of C.