AE 510 - Linear System Theory - Winter 2021

Homework 2

Due Date: Sunday, Jan 24^{th}, 2021 at 11:59 pm

1. **Similarity Transformations**

   - Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   and the equation $y = Ax$ for $x, y \in \mathbb{R}^2$. For each coordinate transformation $T \in \mathbb{R}^{2\times 2}$ shown below, compute the matrix A' such that $y' = A'x'$ when $x = Tx'$ and $y = Ty'$.

(PTS: 0-2) $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, **(PTS: 0-2)** $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

   - Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   and the equation $y = Ax$ for $x, y \in \mathbb{R}^2$. For each coordinate transformation $T \in \mathbb{R}^{2\times 2}$ shown below, compute the matrix A' such that $y' = A'x'$ when $x = Tx'$ and $y = Ty'$.

(PTS: 0-2) $T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, **(PTS: 0-2)** $T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$,

2. **Finding a Nullspace Basis**

   (a) **Basis Derivation**

   Consider a fat matrix $A \in \mathbb{R}^{m\times n}$ $(m < n)$ that is partitioned as $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ with $A_1 \in \mathbb{R}^{m\times m}$ invertible. Show that the columns of $B \in \mathbb{R}^{n\times n-m}$

$$B = \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix}$$

   form a basis for the nullspace of A, $\mathcal{N}(A)$ by performing the following two steps.

   i. **(PTS: 0-2)** Show that any vector $v \in \mathcal{N}(A)$ can be written as $v = Bw$ for some $w \in \mathbb{R}^{n-m}$, ie. v is linear combination of the columns of B (the columns of B span the nullspace).

   ii. **(PTS: 0-2)** Show that the columns of B are linearly independent.

(b) **Computation**

For the following matrices explicitly compute a basis for their nullspaces BY HAND, ie. do not use computational software.

i. **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

ii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

iii. **(PTS: 0-2)**

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{bmatrix}$$

3. **Matrix Rank**

The column rank of a matrix is the number of linearly independent columns. The row rank of a matrix is the number of linearly independent row.

(a) **(PTS: 0-2)** Show that the row rank is less than or equal to the column rank.

(b) **(PTS: 0-2)** Show that the col rank is less than or equal to the row rank.

(Note: The fact that row rank = column rank is a deep and fundamental fact of matrices and linear combinations of vectors. The way to do this proof can be found in the lecture notes `span.pdf` (or on Wikipedia).)

4. **Range and Nullspace**

Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent the range and nullspace of A (and similarly let $\mathcal{R}(A^T)$ and $\mathcal{N}(A^T)$ be the range and nullspace of A^T).

(a) **(PTS: 0-2)** Suppose $y \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$. Show that $x \perp y$, ie. $x^T y = 0$.

(b) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{5\times 10}$. Suppose A has only 3 linearly independent columns (the other 7 are linearly dependent on the first 3). What is the dimension of $\mathcal{R}(A)$? What is the dimension of $\mathcal{N}(A^T)$? What is the dimension of $\mathcal{N}(A)$? What is the dimension of $\mathcal{R}(A^T)$? (It's ok to just state the answers without proof.)

5. **Least Squares and Minimum Norm Solutions**

(a) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{m\times n}$ where $m > n$ (A is "tall") and A has full-column rank (the columns are linear independent). Show that the least squares solution $x = (A^\intercal A)^{-1} A^\intercal y$, minimizes $|y - Ax|^2$, ie. makes Ax as close as possible to y.

(b) **(PTS: 0-2)** Consider $A \in \mathbb{R}^{m \times n}$ where $m < n$ (A is "fat") and A has full-row rank (the rows are linear independent). Let $x = A^\intercal (AA^\intercal)^{-1} y$ and $z \in \mathbb{R}^n$ be any vector such that $y = Az$. Show that $|x| \leq |z|$.

6. **Grammian Rank**

   (PTS: 0-2) Show that $A^T A$ has the same rank as $A \in \mathbb{R}^{m \times n}$. Hint: use the rank-nullity theorem and show that both matrices have identical nullspaces.

7. **Basis for Domain from Nullspace of A and Range of A^T**

   Consider $A \in \mathbb{R}^{m \times n}$ with $m < n$ and full row rank and a matrix $N \in \mathbb{R}^{n \times n-m}$ with full column rank whose columns span the nullspace of A. Suppose we write a vector $x \in \mathbb{R}^n$ as a linear combination of the rows of A and the columns of N, ie.

$$x = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

   for $x_1' \in \mathbb{R}^m$ and $x_2' \in \mathbb{R}^{n-m}$

   (a) **(PTS: 0-2)** Symbollically compute $\begin{bmatrix} A^T & N \end{bmatrix}^{-1}$.

   Hint: Start by checking if $\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} A^T & N \end{bmatrix}^T$...

   (b) **(PTS: 0-2)** Solve for x_1' and x_2' given A,N, and x.

8. **Fundamental Theorem of Linear Algebra Pictures**

   For each of the following matrices draw a picture of the domain (either $\mathbb{R}^2$ or $\mathbb{R}^3$) labeling $\mathcal{R}(A^T)$ and $\mathcal{N}(A)$ and a picture of the co-domain (either $\mathbb{R}^2$ or $\mathbb{R}^3$) labeling the $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$.

(PTS: 0-2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(PTS: 0-2) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(PTS: 0-2) $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$

(PTS: 0-2) $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$