AE510 - Linear System Theory - Winter 2021

Homework 3

<u>Due Date</u>: Sunday, Jan 31^{st}, 2021 at 11:59 pm

1. **Elementary Matrices and Matrix Inverses**

   Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   (a) **(PTS: 0-2)** Compute a sequence of elementary matrices that could be used to row-reduce A to the identity.

   (b) **(PTS: 0-2)** Use this sequence of elementary matrices to compute A^{-1}.

2. **Block Matrix Inversion**

   Consider the block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

   (a) **(PTS: 0-2)** Show that

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

   (b) **(PTS: 0-2)** Show that

$$\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & 0 \\ 0 & F^{-1} \end{bmatrix}$$

   (c) **(PTS: 0-2)** Show that

$$\begin{bmatrix} I & G \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -G \\ 0 & I \end{bmatrix}$$

   (d) **(PTS: 0-2)** Show that

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

   Note: you can do this either by using the first three parts or by showing directly that $M^{-1}M = I$.

3. **Woodbury Matrix Identity**

   Let $M = A + UCV$ where $M, A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times m}$ and let $n > m$.

   (a) **(PTS: 0-2)** What are the dimensions of U and V? Which one is tall and which one is fat?

   (b) **(PTS: 0-2)** Show the Woodbury Matrix Identity

$$M^{-1} = (A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}.$$

4. **Similar Eigenvalues**

   (a) **(PTS: 0-2)** Let $A \in \mathbb{R}^{n \times n}$ and let $T \in \mathbb{R}^{n \times n}$ be any non-singular matrix. Show that the eigenvalues of A are the same as those of $T^{-1}AT$. What are the eigenvectors of $T^{-1}AT$?

   (b) **(PTS: 0-2)** Let $A, B \in \mathbb{R}^{n \times n}$ be invertible matrices. Show that the eigenvalues of AB are the same as those of BA.

5. **Spectral Mapping Theorem**

   Consider a diagonalizable matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a polynomial function $f : R^{n \times n} \to R^{n \times n}$.

   (a) **(PTS: 0-2)** Show that the eigenvectors (left and right) of $f(A)$ are thex same as the eigenvectors of A.

   (b) **(PTS: 0-2)** Show that the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

6. **Cayley-Hamilton Theorem**

   (a) **(PTS: 0-2)** The eigenvalues of a matrix A are roots of its characteristic polynomial, $\chi_A(\lambda) = \det(\lambda I - A)$, ie. $\det(\lambda_i I - A) = 0$ if λ_i is an eigenvalue of A. Show that $\chi_A(A) = \mathbf{0}$ (where $\mathbf{0}$ is a matrix of zeros). Note that when you plug the matrix A into $\chi_A(\cdot)$ you replace each constant with that constant times the identity matrix, ie. $\chi(A) = A^3 - 2A^2 + A - I$. (Hint: use the spectral mapping theorem).

   (b) **(PTS: 0-2)** . Suppose that $\chi_A(s) = \det(sI - A) = s^3 - 2s^2 + s - 1$. Use Cayley-Hamilton to write an expression for A^6 in terms of A^2, A, I.

   (c) **(PTS: 0-2)** . Consider an invertible matrix $A \in \mathbb{R}^{n \times n}$ with characteristic polynomial

$$\chi_A(s) = \det(sI - A) = s^n - \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

   Use the fact that $\chi_A(A) = \mathbf{0}$ to write a polynomial expression for A^{-1}. The polynomial should have degree $n - 1$.

7. **Traces and Determinants**

   The determinant and trace of a diagonal matrix can be computed as follows.

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \qquad \Rightarrow \qquad \det(D) = \prod_i \lambda_i, \quad \mathrm{Tr}(D) = \sum_i \lambda_i$$

The determinant operator has the following two properties.

$$\det(A^{-1}) = \det(A)^{-1}, \qquad \det(AB) = \det(A)\det(B)$$

The trace operator has the following property.

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$$

Assume that $A \in R^{n\times n}$ is diagonalizable and let $\lambda_1, \dots, \lambda_n$ be its eigenvalues. Use the properties of traces and determinants to show that

(a) **(PTS: 0-2)**: $\quad \mathrm{Tr}(A) = \sum_i \lambda_i$

(b) **(PTS: 0-2)**: $\quad \det(A) = \prod_i \lambda_i$