AE 510 - Linear System Theory - Winter 2021

Homework 4

<u>**Due Date**</u>: Sunday, Feb 7^{th}, 2021 at 11:59 pm

1. **Low Rank Matrix Structure**

   Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

   - **(PTS:0-2)** Compute a sequence of elementary matrices $E = E_k \cdots E_1$ where $E_i \in \mathbb{R}^{5\times 5}$ such that

$$A = E^{-1}EA = E^{-1}\underbrace{\begin{bmatrix} I_{3\times 3} & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{EA}$$

     What are the dimensions of each block of EA?
   - **(PTS:0-2)** Use the columns of E^{-1} to write a basis for $\mathcal{R}(A)$ and a basis for $\mathcal{N}(A^T)$. Construct a matrix M whose rows are the basis for $\mathcal{N}(A^T)$. Show that $MA = 0$
   - **(PTS:0-2)** Use the rows of EA to write a basis for $\mathcal{R}(A^T)$ and a basis for $\mathcal{N}(A)$. Construct a matrix N whose columns are the basis for $\mathcal{N}(A)$. Show that $AN = 0$

2. **Complex Numbers and Roots of Unity**

   - **(PTS: 0-2)** Consider the complex number $z = (a + bi)$ with $a = b$, $a, b > 0$, and $z^*z = 4$. Write z in polar form, ie. $z = re^{i\theta}$ and, in the complex plane, draw and label the points

$$z, z^*, z^{-1}, (z^*)^{-1}, 1, e^{i\theta}, e^{-i\theta}$$

     .
   - **(PTS: 0-2)** Consider the vector $V_k^4 \in \mathbb{C}^4$ for $k \in \{0, 1, 2, 3\}$ and the *shift matrix* $S \in \mathbb{R}^{4\times 4}$

$$V_k^4 = \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{4}k} \\ e^{i\frac{2\pi}{4}2k} \\ e^{i\frac{2\pi}{4}3k} \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

     Show that $SV_k^4 = e^{i\frac{2\pi}{4}k}V_k^4$.

- (**PTS: 0-2**) List the complex solutions to the equation $z^4 = 1$, $z \in \mathbb{C}$ (the 4-th roots of unity).
- (**PTS: 0-2**) For each 4-th root of unity given above, plot z^t for $t = 0, 1, 2, 3, 4$ in the complex plane.
- (**PTS: 0-2**) Consider the complex signals $f_k(t) = e^{\frac{2\pi ikt}{8}}$ for $k = 0, 1, 2, \ldots, 7$. Plot $\mathbf{Re}(f_k(t))$ for $t = 0, 1, 2, \ldots, 7$ for $k = 0, 1, 2, \ldots, 7$.

3. **Computing Eigenvalues and Diagonalization**

   Compute eigenvalues and left and right eigenvectors for each of the following matrices. Write out a diagonalization for each matrix.

   (a) **(PTS: 0-2)** Eigenvalues, Eigenvectors, Diagonal form

   $$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

   (b) **(PTS: 0-2)** Eigenvalues, Eigenvectors, Diagonal form

   $$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

   (c) **(PTS: 0-2)** Eigenvalues, Eigenvectors, Diagonal form

   $$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

4. **Orthogonal Eigenvectors**

   Suppose $p_1, p_2 \in R^2$ are linearly independent right eigenvectors of $A \in R^{2\times 2}$ with eigenvalues $\lambda_1, \lambda_2 \in R$ such that $\lambda_1 \neq \lambda_2$. Suppose that

   $$p_1^T p_2 = 0, \qquad |p_1| = 1, \qquad |p_2| = 2$$

   (a) **(PTS: 0-2)** Write an expression for a 2×2 matrix whose rows are the left-eigenvectors of A

   (b) **(PTS: 0-2)** Write an expression for a similarity transform that transforms A into a diagonal matrix.