AE510 - Linear System Theory - Winter 2021

Homework 5

Due Date: Monday, Feb 15^{th}, 2021 at 11:59 pm

1. **Eigenstructures**

   Compute the eigenvalues and eigenvectors, and diagonalize each of the following matrices.

   (a) **(PTS: 0-2)**

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   (b) **(PTS: 0-2)**

   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

   (c) **(PTS: 0-2)**

   (Option 1:) In addition to diagonalizing, choose eigenvectors that are orthogonal and show they're orthogonal for matrix

   $$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

   (Option 2:) In addition to diagonalizing, show the eigenvectors are orthogonal for matrix

   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

2. **Complex Eigenvalues and Eigenvectors**

   Compute eigenvalues and right eigenvectors for each of the following matrices. Write out a diagonalization for each matrix. If the matrix has a pair of complex eigenvalues, then write it in both it's diagonal form

   $$A = \begin{bmatrix} | & | & | \\ \frac{1}{\sqrt{2}}(u - vi) & \frac{1}{\sqrt{2}}(u + vi) & p \\ | & | & | \end{bmatrix} \begin{bmatrix} a - bi & 0 & 0 \\ 0 & a + bi & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & \frac{1}{\sqrt{2}}(w^T + y^T i) & - \\ - & \frac{1}{\sqrt{2}}(w^T - y^T i) & - \\ - & q^T & - \end{bmatrix}$$

and the block diagonal form...

$$A = \begin{bmatrix} | & | & | \\ u & v & p \\ | & | & | \end{bmatrix} \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} - & w^T & - \\ - & y^T & - \\ - & q^T & - \end{bmatrix}$$

for vectors $u, v, p, w, y, q \in \mathbb{R}^3$.

(a) **(PTS: 0-2)** Eigenvalues, Eigenvectors, Diagonal form. **(PTS: 0-2)** Block diagonal form.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

(b) **(PTS: 0-2)** Eigenvalues, Eigenvectors, Diagonal form. **(PTS: 0-2)** Block diagonal form.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) **(PTS: 0-2)** Eigenvalues, Eigenvectors, Diagonal form. **(PTS: 0-2)** Block diagonal form.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

3. **Rotation Matrices and Complex Eigenvectors**

Consider the two rotation matrices

$$R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

(a) **(PTS: 0-2)** Show that R_1 and R_2 commute, ie. $R_1R_2 = R_2R_1$ (Note that most matrices do not commute. 2×2 rotation matrices are an exception.)

(b) **(PTS: 0-2)** Compute the inverse of R_1.

(c) **(PTS: 0-2)** Give a physical interpretation of R_1R_2 and R_1^{-1} related to the angles θ_1 and θ_2.

(d) **(PTS: 0-2)** Consider a 2×2 real matrix A that can be diagonalized as

$$A = \tfrac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ (u - vi) & (u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{bmatrix} \left(\begin{bmatrix} | & | \\ (u - vi) & (u + vi) \\ | & | \end{bmatrix} \frac{1}{\sqrt{2}} \right)^{-1}$$

where $r \in \mathbb{R}_+$ and $u, v \in \mathbb{R}^2$. Show that another valid diagonalization for A is

$$A = \tfrac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ (u' - v'i) & (u' + v'i) \\ | & | \end{bmatrix} \begin{bmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{bmatrix} \left(\begin{bmatrix} | & | \\ (u' - v'i) & (u' + v'i) \\ | & | \end{bmatrix} \frac{1}{\sqrt{2}} \right)^{-1}$$

where $u' = \cos(\phi)u + \sin(\phi)v$ and $v' = -\sin(\phi)u + \cos(\phi)v$ for any angle ϕ.

4. **Nilpotent Matrices**

   (PTS:0-2) Consider the matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

   What is N^2? What is N^3? What is N^4? What is the characteristic polynomial of N? What are the eigenvalues of N?

5. **Continuous vs. Discrete Time Stability**

   (PTS: 0-2) Let $\rho(M)$ represent the set of eigenvalues or *spectrum* of M. Show that if $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \rho(A)$, then $|\mu| < 1$ for all $\mu \in \rho(e^{A\Delta t})$.

6. **Circulant Matrices**

   Consider the *shift* matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

   and the *circulant matrix* (for the vector $\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_4 c_5 \end{bmatrix}$).

$$C = \begin{bmatrix} c_0 & c_5 & c_4 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_5 & c_4 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_5 & c_4 & c_3 \\ c_3 & c_2 & c_1 & c_0 & c_5 & c_4 \\ c_4 & c_3 & c_2 & c_1 & c_0 & c_5 \\ c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \end{bmatrix}$$

   Note that each column of C is the vector c shifted by one element.

   (a) **(PTS: 0-2)** Check that the columns of V are right eigenvectors of P. **(You can just check**

2 of them.)

$$V = \begin{bmatrix} | & \cdots & | \\ V_1 & \cdots & V_6 \\ | & \cdots & | \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{i2\pi(1\times1)}{6}} & e^{\frac{i2\pi(1\times2)}{6}} & e^{\frac{i2\pi(1\times3)}{6}} & e^{\frac{i2\pi(1\times4)}{6}} & e^{\frac{i2\pi(1\times5)}{6}} \\ 1 & e^{\frac{i2\pi(2\times1)}{6}} & e^{\frac{i2\pi(2\times2)}{6}} & e^{\frac{i2\pi(2\times3)}{6}} & e^{\frac{i2\pi(2\times4)}{6}} & e^{\frac{i2\pi(2\times5)}{6}} \\ 1 & e^{\frac{i2\pi(3\times1)}{6}} & e^{\frac{i2\pi(3\times2)}{6}} & e^{\frac{i2\pi(3\times3)}{6}} & e^{\frac{i2\pi(3\times4)}{6}} & e^{\frac{i2\pi(3\times5)}{6}} \\ 1 & e^{\frac{i2\pi(4\times1)}{6}} & e^{\frac{i2\pi(4\times2)}{6}} & e^{\frac{i2\pi(4\times3)}{6}} & e^{\frac{i2\pi(4\times4)}{6}} & e^{\frac{i2\pi(4\times5)}{6}} \\ 1 & e^{\frac{i2\pi(5\times1)}{6}} & e^{\frac{i2\pi(5\times2)}{6}} & e^{\frac{i2\pi(5\times3)}{6}} & e^{\frac{i2\pi(5\times4)}{6}} & e^{\frac{i2\pi(5\times5)}{6}} \end{bmatrix}$$

(b) **(PTS: 0-2)** What are the eigenvalues associated with each eigenvector? Which eigenvectors are conjugate pairs of each other?

(c) **(PTS: 0-2)** Show that the circulant matrix C can be written as

$$C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 + c_4 P^4 + c_5 P^5$$

(d) **(PTS: 0-2)** Show that the columns of V are orthogonal to each other, ie. $V_i^* V_j = 0$ for $i \neq j$ where V_i and V_j are columns of V. What does this say about V^{-1}?

(e) **(PTS: 0-2)** Use the spectral mapping theorem to compute the eigenvalues of C.

(f) **(PTS: 0-2)** Write out a diagonalization of C.