AE510 - Linear System Theory - Winter 2021

Homework 6

<u>**Due Date**</u>: Monday, Feb 22^{nd}, 2021 at 11:59 pm

1. **Vector Fields and Stability**

   For each of the following matrices,

   $$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

   consider the system differential equation

   $$\dot{x} = Ax$$

   - **(PTS: 0-2 (for each))** What are the eigenvalues of A? Is the system stable? Sketch the vector field for each system labeling the eigenvectors and show a sample trajectory.
   - **(PTS: 0-2 (for each))** Convert each differential equation into a discrete time update equation

     $$x^+ = A'x$$

     for a time step of $\Delta t = 0.1$. What are the eigenvalues of A'? How do they relate to the eigenvalues of A?

2. **Second Order System Modeling**

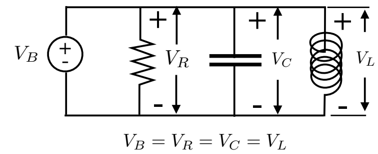
   Spring-mass-damper systems and RLC (resistor-inductor-capacitor) circuits are the two most basic linear 2nd-order systems.

   Modeling in spring-mass-damper systems is based on Newton's 2nd Law, the sum of forces on an object is equal to that object's mass times it's acceleration.

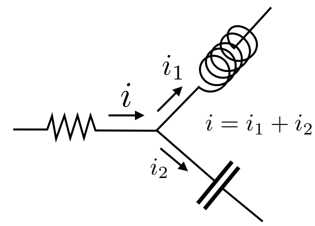
   $$\textbf{Newton's 2nd Law:} \qquad \sum_k F_k = m_i \ddot{x}_i$$

   for forces F_k, mass m_i, and acceleration $\ddot{x}_i$.

   Modeling in RLC circuits is based on Kirchoff's laws, the sum of voltage changes around a loop in a circuit equals zero and the sum of currents at a junction equals zero.



Kirchoff's Voltage Law (KVL): $\sum_{k \in \text{loop}} V_k = 0$

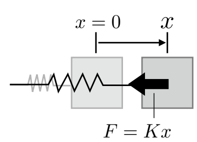
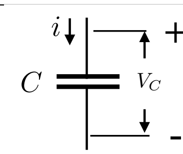
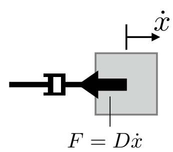
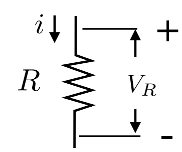
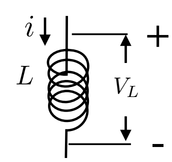


Kirchoff's Current Law (KCL): $\sum_{k \in \text{junc.}} i_k = 0$

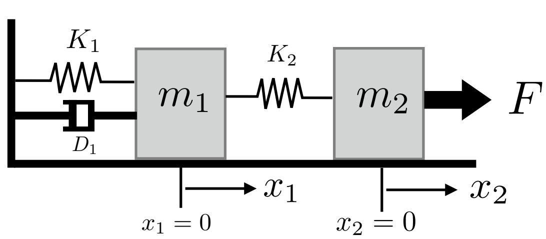
Elements in each system have parallels in the other.

SMD	position (x)	Damper (D)	Spring Const. (K)	Mass (m)	Force (F)
RLC	charge (q)	Resistance (R)	Capacitance (C)	Inductance (L)	Voltage (V)

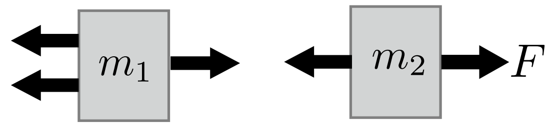
Force rules in spring-mass-damper systems also parallel voltage rules in RLC circuits.

Spring-Mass-Damper			RLC-Circuit		
Law	Equation	Diagram	Law	Equation	Diagram
Hooke's Law	$F = k\Delta x$		**Capacitor**	$\Delta V = Cq$	
Damper	$F = D\dot{x}$		**Ohm's Law**	ΔV = iR	
Newton 2nd Law	$F = m\ddot{x}$		**Inductor**	$\Delta V = L\frac{\partial i}{\partial t}$	

- **Spring-Mass-Damper System**
  Consider the spring-mass-damper system given in the following diagram



(a) **(PTS:0-2)** Label the forces in the free body diagrams for both blocks. Use Newton's 2nd Law - "sum of forces on an object = the object's mass times it's acceleration" to write an equation of motion for each block.

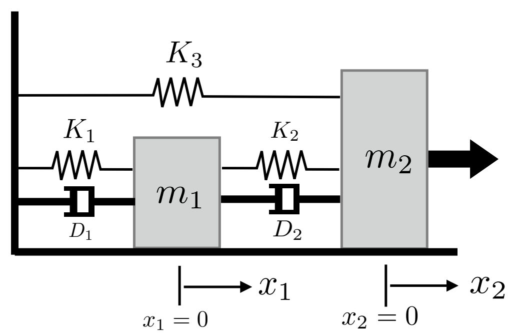


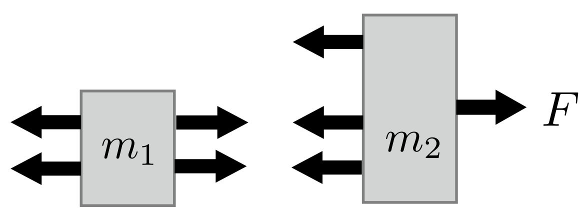
(b) **(PTS:0-2)** Combine the equations of motion into a state space model of the form

$$\dot{x} = Ax + Bu$$

for for $x \in \mathbb{R}^4$, $A \in \mathbb{R}^{4\times 4}$, $B \in \mathbb{R}^{4\times 1}$

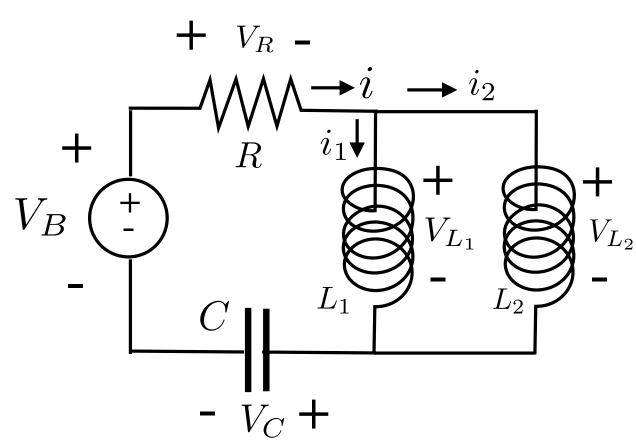
(c) **(PTS:0-4)** Repeat this process for the following system. Draw free body diagrams with forces labeled, apply Newton's 2nd Law, and organize the equations into a state space model for $x \in \mathbb{R}^4$.





- **Resistor-Inductor-Capacitor Circuit**

  Consider the circuit model show in the following diagram



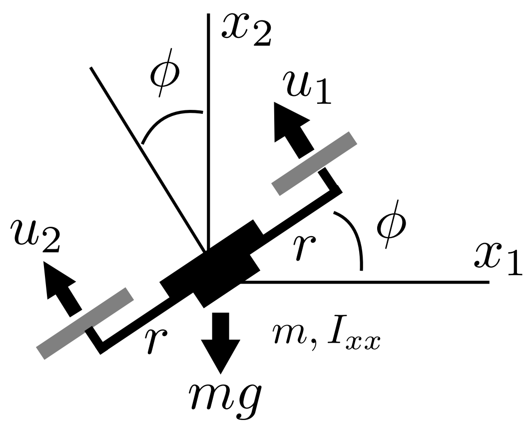
(a) **(PTS:0-2)** Use Kirchoff's voltage laws to write an equation for the change in current through each inductor.

(b) **(PTS:0-2)** Use the facts that $i = i_1 + i_2$ (KCL) and $i = \frac{\partial q}{\partial t}$ to write and equation for $\dot{q}$ in terms of i_1 and i_2.

(c) **(PTS:0-2)** Combine the equations of motion into a state space model of the form $\dot{x} = Ax + Bu$ for $x = [q \ i_1 \ i_2]^T \in \mathbb{R}^3$ with $A \in \mathbb{R}^{3\times3}$, $B \in \mathbb{R}^{3\times1}$, and $u = V_B$.

3. **Simple Drone Model**

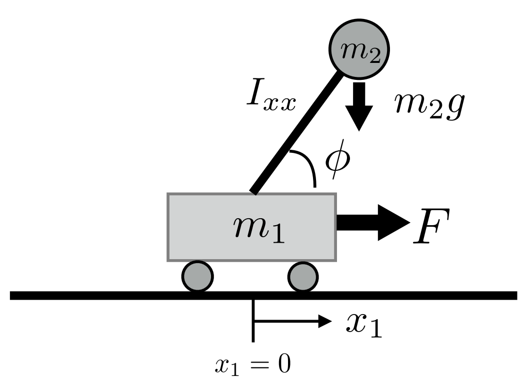
Consider the simple 2D drone model.



(PTS: 0-6) Given mass m, moment of inertia I_{xx}, and arm length r, sum the forces and moments to write a nonlinear dynamics model $\dot{x} = f(x, u)$ for $x = [x_1 \ \dot{x}_1 \ x_2 \ \dot{x}_2 \ \phi \ \dot{\phi}]^T \in \mathbb{R}^6$ and $u = [u_1 \ u_2]^T$.

4. **Inverted Pendulum (EXTRA CREDIT)**

Consider the inverted pendulum model on a cart.



Let m be the mass of the cart, I_{xx} be the moment of inertia of the pendulum, r the length of the pendulum, and F the force on the cart.

(a) **(PTS: 0-6)** Draw free body diagrams of the pendulum and the cart and write force and moment balances to get equations of the position and velocity of the cart $(x_1, \dot{x}_1)$ and the orientation and angular velocity of the pendulum $(\phi, \dot{\phi})$.