AE 510 - Linear System Theory - Winter 2021

Homework 7

Due Date: Monday, Mar 1^{st}, 2021 at 11:59 pm

1. **Integral Solution**

   (PTS:0-4) Show that the solution

   $$x(t) = e^{At}x^0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\ d\tau$$

   satisfies the differential equation and initial condition

   $$\dot{x} = Ax + Bu, \quad x(0) = x^0$$

2. **Minimum Norm Control**

   Consider the discrete update equation

   $$x^+ = Ax + Bu, \quad x[0] = x^0$$

   for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

   - **(PTS: 0-2)** Write the equation for the value of x at the k-th timestep, $x[k]$, in terms of the initial condition x^0 and the control inputs up to time $k-1$, $u[0], \ldots, u[k-1]$.
   - **(PTS: 0-2)** (If your first answer wasn't in this form already), write your previous answer in the form

     $$x[k] = Fx^0 + GU$$

     where $U \in \mathbb{R}^{km}$ is the stacked control vector

     $$U = \begin{bmatrix} u[0] \\ \vdots \\ u[k-1] \end{bmatrix}$$

     What are the matrices F and G in terms of A and B?
   - **(PTS: 0-2)** Consider the problem of driving the system to a particular point x' at time k, ie. computing U such that

     $$x' = x[k] = Fx^0 + GU$$

     Find the minimum norm U such that $x[k] = x'$. Hint: write the above equation in the form $y = GU$ and use the minimum norm solution $U = G^T(GG^T)^{-1}y$. What rank condition on G is required for you to be able to compute this U?

3. **Controllability Matrix Range**

   (PTS:0-4) A result of the Cayley Hamilton theorem ($\chi_A(A) = 0$) is that for a matrix $A^{n\times n}$, for any power $k \geq n$, A^k can be written as

$$A^k = \beta_{n-1}^k A^{n-1} + \beta_{n-2}^k A^{n-2} + \cdots \beta_1^k A + \beta_0^k I$$

   for some coefficients $\beta_{n-1}^k, \ldots \beta_0^k$. Use this fact to show that

$$\mathcal{R}\left(\begin{bmatrix} A^k B & A^{k-1}B & \cdots & AB & B \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} A^{n-1} B & A^{n-2}B & \cdots & AB & B \end{bmatrix}\right)$$

   for any $B \in \mathbb{R}^{n\times m}$ and $k \geq n$.

4. **Controllability Tests**

   Test whether or not the following pairs (A, B) are controllable.

   - **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   - **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   - **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

   - **(PTS: 0-2)**

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. **Non-controllable Systems**

   Each of the following systems of the form

$$\dot{x} = Ax + Bu, \quad x(0) = x^0$$

   is not controllable. For each system find an eigenvector direction that evolves on it's own completely independent of the control input. Hint: To start, apply a coordinate transformation $x = Pz$ that diagonalizes the A matrix and find the coordinate z_i that is unaffected by the control input.

- **(PTS: 0-2)**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- **(PTS: 0-2)**

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- **(PTS: 0-2)**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

- **(PTS: 0-2)**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$