AE 510 - Linear Systems Theory - Winter 2021

Homework 8

Due Date: Monday, Mar 8^{th}, 2021 at 11:59pm

1. **Controllability/Observability: Coordinate Invariance**

   Consider a dynamical system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx + Du$$

   The system is controllable and observable. Show that under the coordinate transformation $x = Tz$

   - **(PTS: 0-2)**. The system is still controllable in the z-coordinates.
   - **(PTS: 0-2)**. The system is still observable in the z-coordinates.

2. **Controllability/Observability Tests**

   Consider the dynamical system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

   where $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$, $D \in R^{1\times 1}$ and A is diagonalizable with right and left eigenvectors the columns and rows of P and Q respectively

$$P = \begin{bmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{bmatrix}, \qquad Q = P^{-1} = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \tag{1}$$

   - **(PTS: 0-2)** Suppose there exists a left eigenvector of A, $q^T \in R^n$ such that $q^T B = 0$. Show that the system is not controllable.
   - **(PTS: 0-2)** Suppose there exists a right eigenvector of A, $p \in R^n$ such that $Cp = 0$. Show that the system is not observable.

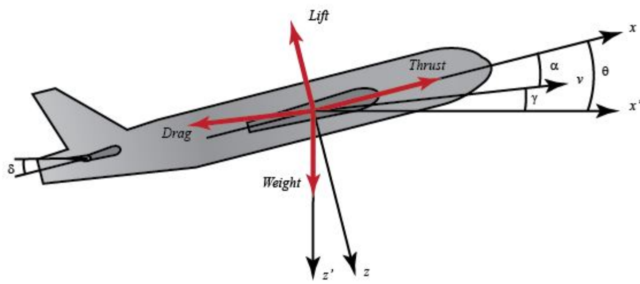
3. **Feedback Control: Eigenvalue Placement**

   For the next few problems consider the aircraft pitch model with the form

$$\dot{x} = Ax + Bu, \quad x(0) = x^0$$
$$y = Cx + Du$$

   with $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{o\times n}$, and $D \in R^{o\times m}$.

   - **Aircraft Pitch**



System Parameters

α = angle of attack	q = pitch rate
θ = pitch angle	δ = elevator deflection angle
$\mu = \frac{\rho S \bar{c}}{4m}$	ρ = air density
S = area of wing	$\bar{c}$ = mean chord length
m = aircraft mass	$\Omega = \frac{2U}{\bar{c}}$
U = equilibrium flight of speed	C_T = Coefficient of Thrust
C_D = Coefficient of Drag	C_L = Coefficient of Lift
C_W = Coefficient of Weight	C_M = Coefficient of Pitch Moment
γ = Flight path angle	$\sigma = \frac{1}{1+\mu C_L}$ = constant
i_{yy} = normalized moment of inertia	$\eta = \mu\sigma C_M$ = constant

Equations of Motion:

$$\dot{\alpha} = \mu\Omega\sigma\left[-\left(C_L + C_D\right)\alpha + \frac{1}{\left(\mu - C_L\right)}q - \left(C_W \sin\gamma\right)\theta + C_L\right]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\left[\left[C_M - \eta\left(C_L + C_D\right)\right]\alpha + \left[C_M + \sigma C_M\left(1 - \mu C_L\right)\right]q + \left(\eta C_W \sin\gamma\right)\delta\right]$$

$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta], \qquad x[0] = \begin{bmatrix} \frac{\pi}{16} \\ -\frac{\pi}{8} \\ \frac{\pi}{12} \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$

Follow the steps given to design a feedback gain matrix K that stabilizes the closed-loop system matrix $A + BK$

(a) **(PTS: 0-2)** Compute the characteristic polynomial of A.

$$\det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$$

Select (distinct) desired eigenvalues $\lambda_1, \ldots, \lambda_n$ for the closed loop system $A + BK$ so that the closed loop system will be stable. (If you want eigenvalues that will make the closed loop system $A + BK$ well conditioned you can use the `lqr` command in Matlab or `control.lqr` in

Python to design the optimal LQR gains K_{lqr} and then compute the eigenvalues of $A+BK_{\text{lqr}}$ for the desired eigenvalues. You can use the cost matrices $Q = I_{3\times 3}$ and $R = 1$. Of course, doing this and then going back and doing pole placement would be silly if this wasn't a homework problem, but whatever.)

Compute the desired characteristic polynomial for $A + BK$ using the formula

$$\det(\lambda I - (A + BK)) = \Pi_i(\lambda - \lambda_i) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0$$

(b) **(PTS: 0-2)** If the system is controllable, compute a coordinate transformation $x = Tz$ such that the system in the z coordinates is in *controllable canonical form*

$$\dot{z} = \bar{A}z + \bar{B}u$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Use the fact that if T exists, then the controllability matrix in the two different coordinates are related by

$$\begin{bmatrix} \bar{A}^{n-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix} = T^{-1} \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \tag{2}$$

(c) **(PTS: 0-2)** Compute the gain matrix $\bar{K}$ such that $\bar{A} + \bar{B}\bar{K}$ has the desired characterisitic polynomial. $\lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0$.

(d) **(PTS: 0-2)** Compute the feedback gain matrix K so that the closed loop system matrix $A + BK$ has the desired characteristic polynomial using $\bar{K}$ and T.

(e) **(PTS: 0-2)** Check that the closed-loop system matrix $A + BK$ is stable.

4. **Observability and Least Squares**

For the aircraft pitch system perform the following steps

(a) **(PTS: 0-2)** Check whether or not the closed-loop system is observable.

(b) **(PTS: 0-2)** Simulate the trajectory forward using the feedback gain you computed in the previous problem from the initial condition given in the model for 100 time steps with a time step size of $\Delta t = 0.01$ seconds. At each time t, compute the output with added sensor noise using the equation

$$y(t) = Cx(t) + v(t), \qquad v(t) \sim \mathcal{N}(0, 1)$$

where $v(t)$ is a scalar Gaussian random variable with normal distribution $\mathcal{N}(0, 1)$ with mean 0 and variance 1. You can find Matlab and python functions to sample from a normal distribution to compute $v(t)$ at each time step. (Note that since θ is measured in radians, a noise covariance of 1 is quite large.)

(c) **(PTS: 0-2)** If the system is observable, use the output trajectory $y(0), \dots, y(T)$ and the method of least squares to compute the initial condition $x(0)$. Try different values for the final time T including $T = 3$ (the minimum final time possible), $T = 10$, and $T = 100$. Which version gives you the best estimate of $x(0)$?

5. **Observer Design and the Separation Principle**

For the two systems above

(a) **(PTS: 0-2)** Write down the estimator dynamics for a state estimate $\hat{x} \in R^n$ given a linear system of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$, and $D \in R^{1\times 1}$.

Note that these should include the observer gain $L \in \mathbb{R}^{n\times 1}$ times the output estimate error.

(b) **(PTS: 0-2)** Write down the joint dynamics of the true state $x \in R^n$ and the estimator state $\hat{x} \in R^n$.

(c) **(PTS: 0-2)** Write down the coordinate transformation $T \in R^{2n\times 2n}$ such that

$$\begin{bmatrix} x \\ e \end{bmatrix} = T \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

where $e \in R^n$ is the error in the state estimate $e = \hat{x} - x$. Use this coordinate transformation to transform the dynamics from the previous part into dynamics for $\begin{bmatrix} x \\ e \end{bmatrix}$.

(d) **(PTS: 0-2)** Show that the stability of the joint dynamics depends on the stability of $A+BK$ and $A + LC$ separately.

(e) **(PTS:0-2)** For the aircraft pitch model given above, design observer gains $L \in R^{n\times 1}$ such that the matrix $A + LC$ is stable. If you want you can use the pole placement method from the previous problem, but you can also just use the Matlab `place` command (or the `lqr` command) which should be significantly faster.

(f) Simulate the joint state-error system using the observer gain L computed above and the control input $u = K\hat{x} + r$ where K is the feedback gain computed above and r is each of the two following reference signals. Use the initial conditions $x(0)$ given with the dynamics and the initial state estimate $\hat{x}(0) = 0$.

- **(PTS: 0-2)** $r = 1$
- **(PTS: 0-2)** $r = \gamma \sin(\omega t)$ (Pick a γ and ω you find interesting.)

(g) **(PTS: 0-2)** Plot the state, error, and control trajectories for each case.