AE 510 - Linear Systems Theory - Winter 2021

Homework 9

Due Date: OPTIONAL

1. **Laplace Transform**

   The Laplace transform of a function $f(t)$ is given by

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\, dt$$

   (a) **Choose three** of the following Laplace Transforms to compute:

   - **(PTS: 0-2)** Delta function: $\mathcal{L}\left(\delta(t)\right) = ?$
   - **(PTS: 0-2)** Differentiation: $\mathcal{L}\left(\dot{f}(t)\right) = ?$
   - **(PTS: 0-2)** Integration: $\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = ?$
   - **(PTS: 0-2)** Frequency Shift: $\mathcal{L}\left(e^{at}f(t)\right) = ?$
   - **(PTS: 0-2)** Convolution: $\mathcal{L}\left(\int_0^t g(t-\tau)f(\tau)\, d\tau\right) = ?$

   (b) **(PTS: 0-2)** Find the Laplace transform of $y(t) = f(t) * h(t)$ where $*$ is the convolution operator.

$$h(t) = e^{-t}, \quad t \geq 0$$

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t \leq 2 \\ 0, & 2 < t \end{cases}$$

   (c) **(PTS: 0-2)** Obtain an approximation of $Y(s)$ from part (b) in the following form using the first-order Padé approximation.

$$Y_{\text{approx}}(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}, \quad m \leq n$$

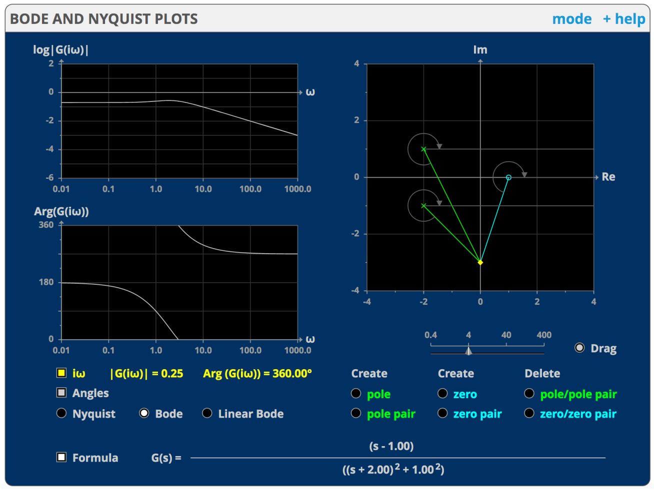
   Note: A Padé approximant is the best approximation of a function by a rational function of given order. The first-order Padé approximation of a time delay τ is

$$e^{-s\tau} \approx \frac{1 - \frac{\tau}{2}s}{1 + \frac{\tau}{2}s}.$$

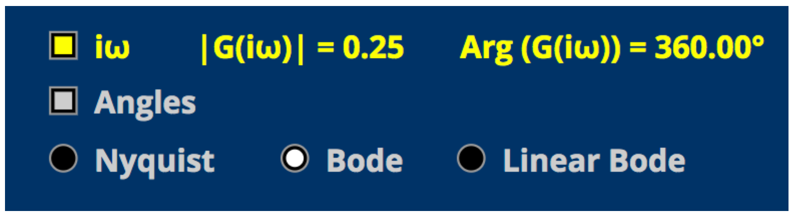
2. **Bode and Nyquist Plots**

   Use the tool in the link below to visualize the Bode plots for the transfer functions listed below.

   https://mathlets.org/mathlets/bode-and-nyquist-plots/



Make sure to try out the $i\omega$ and **Angles** options.



For each case, **take at least one screenshot and comment** on the behavior of the transfer function and how it is represented in the Bode plots.

Note: The goal is to get intuition on how poles and zeros affect transfer function behavior. Make sure you spend as much or more time playing around with it as you do taking screenshots.

(a) **(PTS: 0-2)** One real pole. Vary λ along the real axis. Specifically note what happens when $\lambda = 0$.

$$G(s) = \frac{1}{s - \lambda}$$

(b) **(PTS: 0-2)** A pair of complex poles. Vary the poles and specifically note what happens when they cross the $j\omega$-axis.

$$G(s) = \frac{1}{(s - \lambda_1)(s - \lambda_2)}$$

(c) **(PTS: 0-2)** One real pole and one real zero. Vary the zero z along the real axis. In particular note, the high frequency behavior of the transfer function.

$$G(s) = \frac{s - z}{s - \lambda}$$

(d) **(PTS: 0-2)** A pair of complex poles and a real zero. Vary the zero z along the real axis. In particular note what happens to the phase when the zero crosses the $j\omega$-axis.

$$G(s) = \frac{s - z}{(s - \lambda_1)(s - \lambda_2)}$$

(e) **(PTS: 0-2)** A pair of complex poles and a pair of complex zeros. Vary the zeros. Note what happens when the zeros cross the $j\omega$-axis.

$$G(s) = \frac{(s - z_1)(s - z_2)}{(s - \lambda_1)(s - \lambda_2)}$$

3. ***Z*-transforms**

(PTS: 0-2) Consider a function $f(t)$ and consider a new function $\bar{f}(t)$ obtained by sampling $f(t)$ every Δt seconds.

$$\bar{f}(t) = \sum_{k=0}^{\infty} f(t)\delta(t - \Delta tk) = \begin{cases} f(\Delta tk) & ; \text{ when } t = k\Delta t \text{ for } k = 0, 1, 2, \ldots \\ 0 & ; \text{ otherwise} \end{cases}$$

Compute the Laplace transform of $\bar{f}(t)$ and show that you obtain the z-transform

$$\mathcal{L}(\bar{f}) = \sum_{k=0}^{\infty} f(\Delta tk) z^{-k}$$

where z is defined as $z = e^{s\Delta t}$.

4. **Transfer Functions**

Consider the continuous time linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

with $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{o \times n}$, and $D \in R^{o \times m}$.

(a) **(PTS: 0-2)** Compute the Laplace transform of the output $y(t)$, ie. $\mathcal{L}(y(t)) = Y(s)$. Your solution should be in terms of the $A, B, C, D, x_0, U(s)$, where $U(s)$ is the Laplace transform of $u(t)$.

(b) **(PTS: 0-2)** Assuming A is diagonalizable, expand out your previous answer to show that

$$Y(s) = \sum_i \left(\tfrac{1}{s-\lambda_i} C p_i q_i^T x_0 \right) + \left[\sum_i \left(\tfrac{1}{s-\lambda_i} C p_i q_i^T B \right) + D \right] U(s)$$

where p_i and q_i are the right and left eigenvectors for eigenvalue λ_i.