LECTURE 1:

LINEAR ALGEBRA Review: Part 1

vector spaces: "a vector space V over a field F"

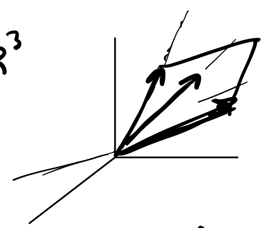
mathematical defn V : addition operation $v_1 \in V$, $v_2 \in V$ $v_1 + v_2$

scalar multiplication $v_1 \in V$, $\alpha \in F$ αv_1

closed under addition & scaler multiplication $v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in F$ $\alpha_1 v_1 + \alpha_2 v_2 \in V$

scale vectors, add them — stay vectors space

$\mathbb{R}^3$



Dot products / Inner products

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad y \in \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

Algebraic Defn

$$x \cdot y = \langle x, y \rangle = x^T y = \sum_i x_i y_i$$

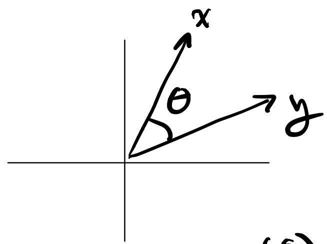
"dot product" "inner product" $\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

length vector / (2-norm)

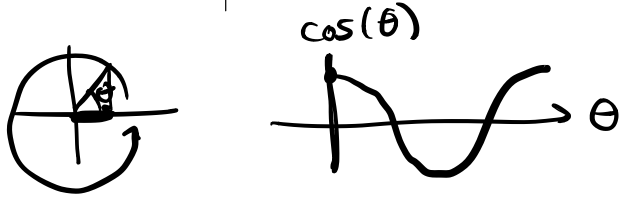
$$|x| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{x \cdot x} = \sqrt{x^T x} \quad \Leftarrow \text{use 1}$$

Geometric defn of inner product

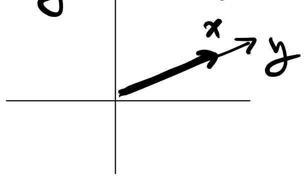
$$x^T y = |x||y| \cos(\theta)$$



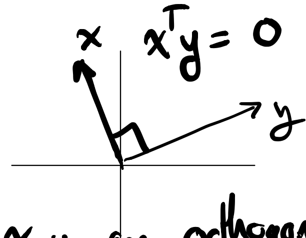
x^Ty : encodes how much x & y lie in the same direction



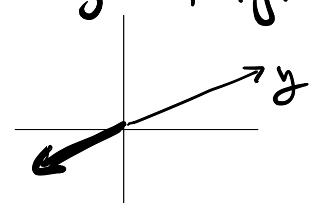
$x^Ty = |x||y|$



$x^Ty = 0$



$x^Ty = -|x||y|$

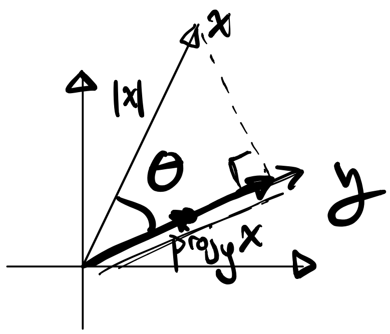


x, y are orthogonal
perpendicular

$x^Ty = 0 \quad x \perp y$

Computing Projections

"compute projection of x onto y."



$\text{proj}_y x = \underbrace{|x|\cos\theta}_{\text{scalar}}$ (unit vector in y direction)

unit vector: length 1. $\frac{y}{|y|}$

$= |x|\cos\theta \, \frac{y}{|y|}$

Dot product version:

$$\boxed{\text{proj}_y x = \frac{1}{|y|^2} y \, y^T x}$$

$\leftrightarrow |x||y|\cos\theta$

$\rightarrow |x| \frac{1}{|y|} \cos\theta \; y$

Norms

$\|x\|_2 = \sqrt{x^T x}$ $\Longleftarrow$ 2-norm or Euclidean norm.

$\|x\|_2 = \left(\sum_i |x_i|^2\right)^{1/2}$ Technically norm or length

$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$

$p = 1, \dots, \infty$

$\|x\|_1 = \sum_i |x_i|$

$\|x\|_2 = \left(\sum_i |x_i|^2\right)^{1/2}$

$\vdots$

$\|x\|_\infty = \left(\sum_i |x_i|^\infty\right)^{1/\infty} \longrightarrow \max_i |x_i|$

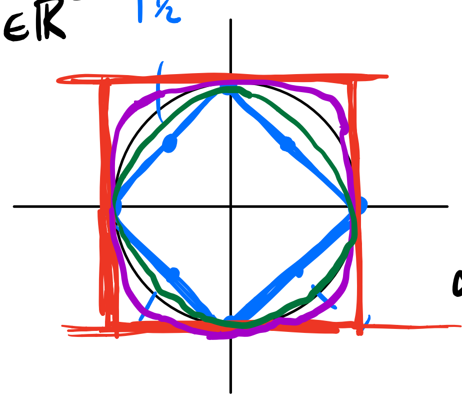
$x \in \mathbb{R}^n$ & $\alpha \in \mathbb{R}$

- $|x+y| \le |x| + |y|$ triangle inequal
- $|\alpha x| = |\alpha||x|$
- $|x| = 0 \Rightarrow x = 0$

Draw p-norm balls

Unit ball...

$x \in \mathbb{R}^2$ $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$



2 norm • $1 = \|x\|_2 = \sqrt{x_1^2 + x_2^2}$

1 norm • $\boxed{1 = \|x\|_1 = |x_1| + |x_2|}$

∞-norm • $\boxed{1 = \max_i |x_i| = \max |x_1|, |x_2|}$

p-norm • $1 = \left(|x_1|^p + |x_2|^p\right)^{1/p}$

$2 < p < \infty$

$1 < p < 2$

Matrix Multiplication

$A \in \mathbb{R}^{m\times n}$ $\quad B \in \mathbb{R}^{n\times p}$ $\quad$ inner dims of A & B match

$AB \in \mathbb{R}^{m\times p}$

$$AB = \begin{bmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn}\end{bmatrix}\begin{bmatrix} b_{11} & b_{1p}\\ b_{n1} & b_{np}\end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+\cdots+a_{1n}b_{n1} & & a_{11}b_{11}+\cdots+a_{11}b_{1p} \\ & [\;,\;] & \\ a_{m1}b_{11}+\cdots+a_{mn}b_{n1} & & a_{m1}b_{1p}+\cdots+a_{mn}b_{np}\end{bmatrix}$$

$$[AB]_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj} = \sum_k a_{ik}b_{kj}$$

Block Matrix Multiplication:

$A_{11}\in\mathbb{R}^{m_1\times n_1}$ $\quad A_{1n}\in\mathbb{R}^{m_1\times n_n}$

$$A = \begin{bmatrix} A_{11} & & A_{1n} \\ & & \\ & & \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & & B_{1p} \\ & & \\ B_{n1} & & \end{bmatrix}$$

General case

$$AB = \begin{bmatrix} A_{11} & & A_{1n} \\ & & \\ A_{m1} & & A_{mn}\end{bmatrix}\begin{bmatrix} B_{11} & & B_{1p} \\ & & \\ B_{n1} & & \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}+\cdots+A_{1n}B_{1n} & \cdots \\ A_{m1}B_{11}+\cdots+A_{mn}B_{n1} & \cdots \end{bmatrix}$$

$$AB = \begin{array}{c c} & \begin{array}{cc} 2 & 8 \end{array} \\ \begin{array}{c} 7 \\ 3 \end{array} & \begin{bmatrix} A_{11}^{7\times 2} & A_{12}^{7\times 8} \\ A_{21}^{3\times 2} & A_{22}^{3\times 8} \end{bmatrix} \end{array} \begin{array}{c c} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} 2 \\ 8 \end{array} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array}$$

$p_1 + p_2 = 10$

$A \in \mathbb{R}^{10\times 10}$

$B \in \mathbb{R}^{10\times 10}$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

$A_{11} = [\cdots ; \cdots ; \cdots]$

$A_{12}\ [\cdots ; \cdots ; \cdots]$

$A_{21}\ [\cdots ; \cdots ; \cdots]$

$A_{22}\ [\cdots ; \cdots ; \cdots]$

$A = [A_{11}, A_{12} ; A_{21}, A_{22}]$

2 specific cases AB

$m \times p$ dot products

$$AB = \begin{bmatrix} - \bar{A}_1^T - \\ \vdots \\ - \bar{A}_m^T - \end{bmatrix} \begin{bmatrix} | & & | \\ B_1 & \cdots & B_p \\ | & & | \end{bmatrix} = \begin{bmatrix} \bar{A}_1^T B_1 & & \bar{A}_1^T B_p \\ \bar{A}_m^T B_1 & & \bar{A}_m^T B_p \end{bmatrix}$$

A as rows & B as cols

$$[AB]_{ij} = \bar{A}_i^T B_j$$

$$AB = \begin{array}{c} \begin{matrix} 1 & \cdots & 1 \end{matrix} \\ \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \begin{bmatrix} - \bar{B}_1^T - \\ \vdots \\ - \bar{B}_n^T - \end{bmatrix} = A_1 \bar{B}_1^T + \cdots + A_n \bar{B}_n^T$$

$$\underbrace{m \begin{bmatrix} A_1 \end{bmatrix} 1 [- \bar{B}_1^T -]}_{m \times p} \qquad \underbrace{m \begin{bmatrix} A_n \end{bmatrix} 1 [- \bar{B}_n^T -]}_{m \times p}$$

sum of n $m \times p$ matrices

$A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - \bar{A}_1^T - \\ - \bar{A}_m^T - \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} \overset{\sum \bar{A}_{1i} x_i}{\bar{A}_1^T x} \\ \vdots \\ \bar{A}_m^T x \end{bmatrix} = \begin{bmatrix} [- \bar{A}_1^T -][x] = [\ ] \\ [- \bar{A}_m^T -][x] = [\ ] \end{bmatrix} \qquad A = \begin{bmatrix} - \bar{A}_1^T - \\ \vdots \\ - \bar{A}_m^T - \end{bmatrix}$$

$$\boxed{\begin{aligned} Ax &= \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A_1 x_1 + \cdots + A_n x_n \\ &= \begin{bmatrix} A_1 \end{bmatrix} x_1 + \cdots + \begin{bmatrix} A_n \end{bmatrix} x_n \end{aligned}}$$

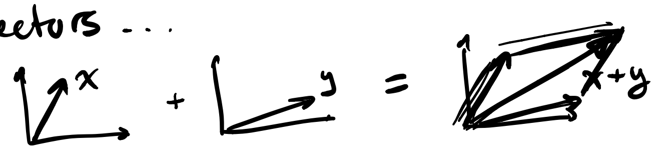
$A = [A_1 \cdots A_n]$

really important

Linear combinations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

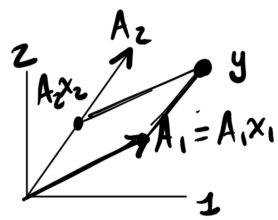
Add vectors ...



$A_1 \in \mathbb{R}^2$

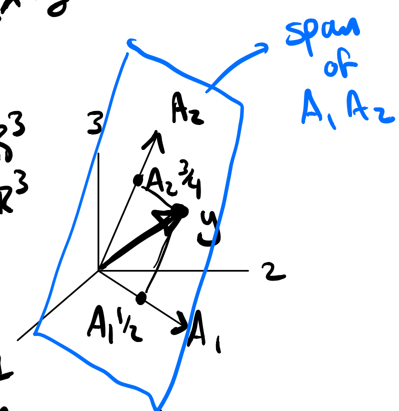
$A_2 \in \mathbb{R}^2$

$y \in \mathbb{R}^2$



$A_1 \in \mathbb{R}^3$

$A_2 \in \mathbb{R}^3$



$$y = A_1 x_1 + A_2 x_2$$

$x_1 = 1$ $\quad x_2 = 1/2$

$$y = A_1 x_1 + A_2 x_2$$

$x_1 = 1/2$ $\quad x_2 = 3/4$

"y is a lin comb of A_1 & A_2"

set of vectors $A_1, \dots, A_n$

$$\text{span}(\{A_1, \dots, A_n\}) = \{ y \mid y = A_1 x_1 + \dots + A_n x_n \quad x \in \mathbb{R}^n \}$$

$$A = [A_1 \cdots A_n]$$

$$\text{span of cols of } A = \{ y \mid y = Ax, \ \underline{x \in \mathbb{R}^n} \}$$

such that.

range(A)

$$\mathcal{R}(A) = \text{span of cols}$$

Ex.

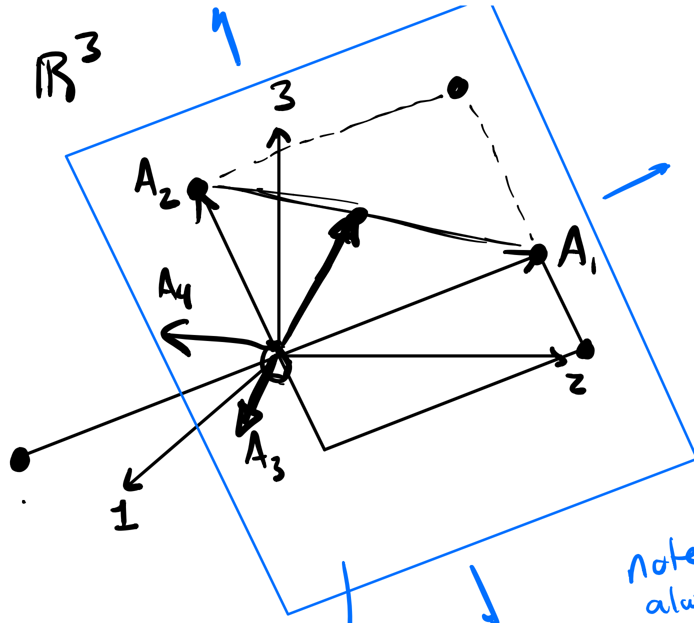
$A = [A_1 \; A_2]$

$y = Ax$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad y = [A_1 A_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A_1$

$x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \; x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$x = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$



note always pass through origin

span $\{[A_1 A_2]\}$ =

$\mathcal{R}(A)$ =

all lin combs of A_1 A_2

Span of $[A_1 \, A_2 \, A_3]$

$A_3 \perp$ to $\mathcal{R}([A_1 A_2])$

Span of $[A_1 A_2 A_3]$

$A_3 = \frac{1}{2} A_1 + \frac{1}{2} A_2$

$\mathcal{R}([A_1, A_2]) = \mathcal{R}([A_1 A_2 A_3])$

if $A_3 \perp \mathcal{R}([A_1 A_2])$

$\mathcal{R}([A_1 A_2 A_3 A_4]) = \mathcal{R}([A_1 A_2 A_3])$

Linear Dependence:

y lin dep on $A_1, \dots, A_n$ if $y \in \mathcal{R}([A_1 \cdots A_n])$

"can construct y as a lin comb of cols of $A = [A_1 \cdots A_n]$"

$\exists x$ s.t. $y = Ax$

there exists

Linear Independent = not lin dependent

y is lin ind. from $A_1 \cdots A_n$ if.

$\nexists\, x$ s.t. $y = Ax$

A set of vectors $A_1, \cdots, A_n$ is lin dep.
if one of them is lin dep on the others...

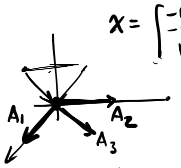
$A_i \in \text{span}\{A_j\}_{j \neq i}$

$$\boxed{\begin{array}{l}\text{if } A_1, \cdots, A_n \text{ lin dep. } A = [A_1 \cdots A_n] \\ \Rightarrow \exists\, x \neq 0 \text{ s.t. } Ax = 0\end{array}}$$

$$A_1 = A_2 x_2 + \cdots + A_n x_n$$

$$[A_1 \cdots A_n] \begin{bmatrix} -1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = -A_1 + A_2 x_2 + \cdots + A_n x_n = 0$$

$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 1 \\ -1 + 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = -x_3$
$x_2 = -x_3$
$0 = 0$

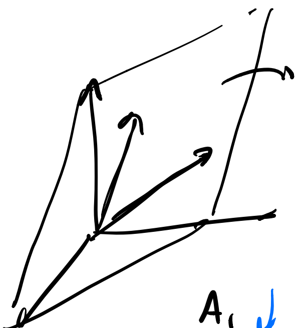
$x_3 = 1$
$x_1 = -1$
$x_2 = -1$

$A_1 \cdots A_n$ is lin ind.
if no vector is lin dep on the others.

$$\boxed{\begin{array}{l}\text{if } A_1 \cdots A_n \text{ lin ind} \quad A = [A_1 \cdots A_n] \\ \text{if } Ax = 0 \Rightarrow x = 0\end{array}}$$

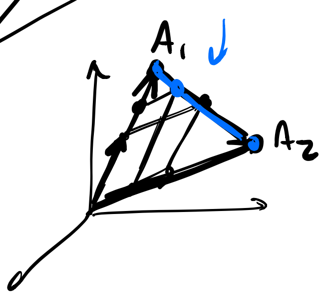
lin comb of cols of $A \in \mathbb{R}^{m \times n}$ $\quad \{ y \in \mathbb{R}^m \mid y = Ax, \; x \in \mathbb{R}^n \}$

convex comb of cols of A $\quad \{ y \in \mathbb{R}^m \mid y = Ax \quad x \in \mathbb{R}^n \quad x \geq 0 \quad \mathbb{1}^T x = 1 \}$



lin comb

$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \mathbb{1}^T x = [1 \cdots 1] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_i x_i$



$A = [A_1 \; A_2] \qquad A \in \mathbb{R}^{3 \times 2}$

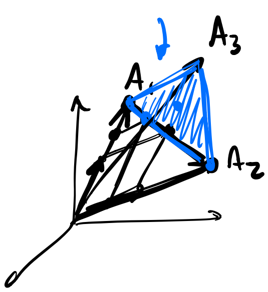
$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$

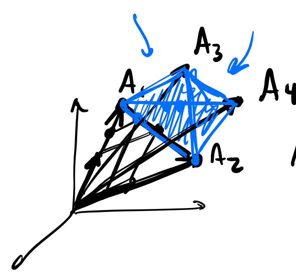
$x_1 \geq 0 \qquad x_1 + x_2 = 1$

$x_2 \geq 0$

x_1	x_2
1	0
0	1
1/2	1/2
3/4	1/4



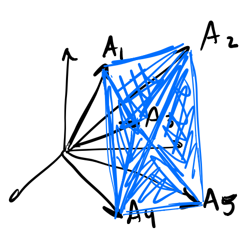
$A = [A_1 A_2 A_3]$



$x = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

$A = [A_1 A_2 A_3 A_4]$

$x_i \geq 0 \qquad \sum_i x_i = 1$



$A = [A_1 \; \cdots \; A_n]$

$\mathcal{A}$ convex if $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{A}$ $\quad (1-\alpha)A_1 + \alpha A_2 \in \mathcal{A}$

$0 \le \alpha \le 1$

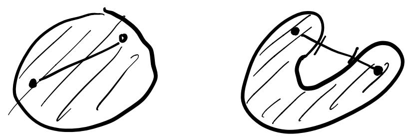
$A_1x_1 + A_2x_2$

s.t. $x_1 \ge 0$

$x_2 \ge 0$

$x_1 + x_2 = 1$

$\alpha = x_2$



convex $\qquad$ not convex

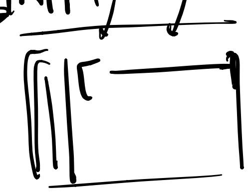
$\to A_1x_1 + A_2x_2 \qquad x_1 + x_2 = 1$

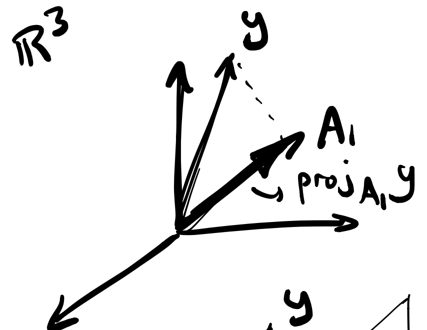
$A_1 + A_1(x_1 - 1) + A_2x_2 \qquad x_2 = 1 - x_1$

$A_1 + (A_2 - A_1)x_2 \to 0 \le x_2 \le 1$

Projections

$$\text{proj}_{A_1} y = \frac{1}{\|A_1\|^2} \overbrace{A_1 A_1^T}^{3\times3 \text{ matrix}} y$$

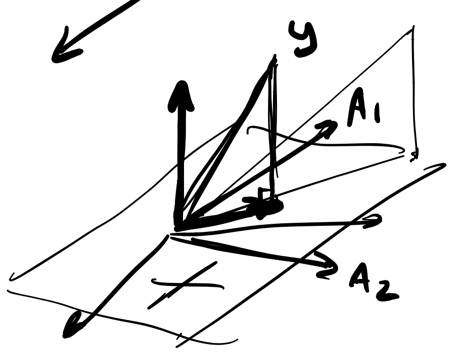


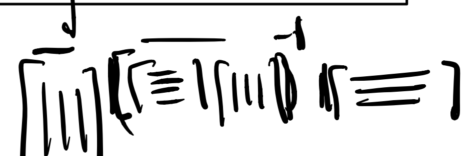


$A = [A_1 \; A_2]$

$(A_1^T A_1)^{-1}$

$$\boxed{\text{proj}_A y = A(A^TA)^{-1}A^T y}$$

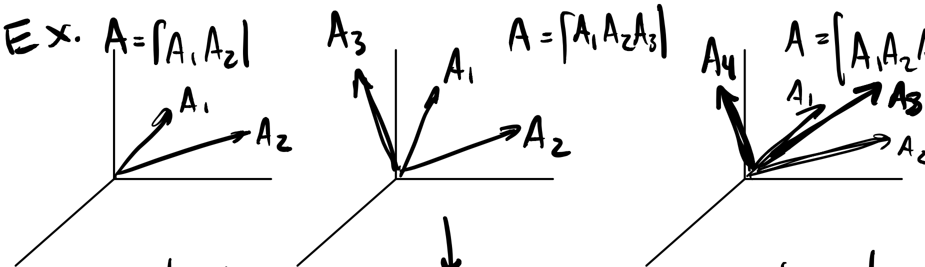




Basis for a vector space

A Basis set of vectors that are
- lin ind.
- span the whole vector space

Ex. $A=[A_1\ A_2]$ $\qquad$ $A=[A_1\ A_2\ A_3]$ $\qquad$ $A=[A_1\ A_2\ A_3\ A_4]$



not a basis for $\mathbb{R}^3$
doesn't span
"too few basis vectors"

basis for $\mathbb{R}^3$

not a basis for $\mathbb{R}^3$
vectors not lin ind.
too many basis vectors

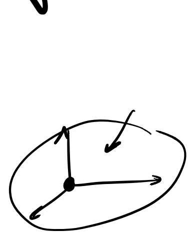
dimension of a space = # of basis vectors

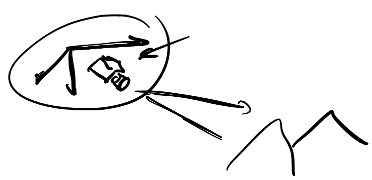
standard basis in $\mathbb{R}^n$

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\cdots\begin{bmatrix}0\\\vdots\\1\end{bmatrix}$$

basis as a "set of coordinate vectors"

"change of basis" = "change of coords"





Coordinates (with respect to a basis)

coordinates of y are coeffs you multiply the basis vectors by to get y.

Basis vectors $A_1 \cdots A_n$ for $\mathbb{R}^n$ $A = [A_1 \cdots A_n]$

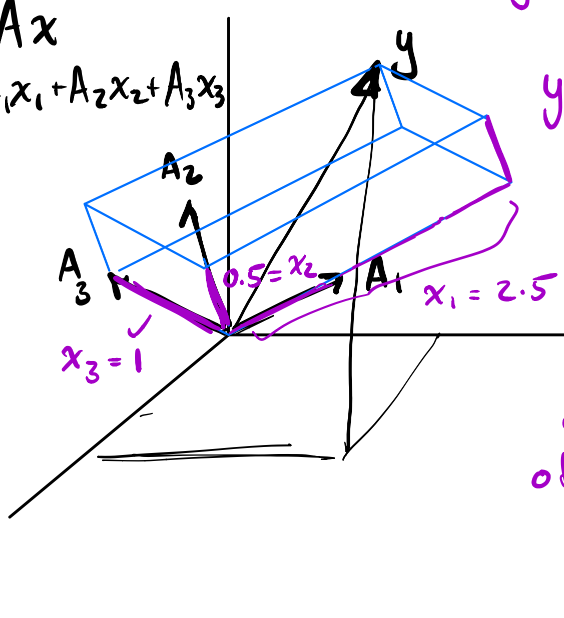
$$\underset{\substack{\uparrow \\ \text{vector}}}{y} = Ax = A_1 x_1 + \cdots + A_n x_n$$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ coordinates of y w.r.t the basis $A = [A_1 \cdots A_n]$

if A is standard basis $A = \begin{bmatrix} 1 & & 0 \\ 0 & \cdots & \vdots \\ 0 & & 1 \end{bmatrix}$ → confusing cause it's simple

$y = Ax = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Ex. $y = Ax$
$= A_1 x_1 + A_2 x_2 + A_3 x_3$



$y = A_1\, 2.5 + A_2\, 0.5$
$+ A_1\, 1$

$y = [A_1 A_2 A_3] \begin{bmatrix} 2.5 \\ 0.5 \\ 1 \end{bmatrix}$

$x = \begin{bmatrix} 2.5 \\ 0.5 \\ 1 \end{bmatrix}$

coordinates of y w.r.t. A

$y = Ax$ compute A^{-1} $y = Ax$

$x = A^{-1}y$ $A^{-1}y = x$

computing A^{-1} is closely related to solving $\leftarrow y = Ax$

undergrad $y = Ax$

Gaussian Elimination = row reduction

$$I = \begin{bmatrix} 1 & & 0 \\ 0 & \ddots & \vdots \\ 0 & & 1 \end{bmatrix}$$

$$[A \mid y] \overset{\text{row reduce}}{\sim} [I \mid x]$$

$$[A \mid I] \overset{\text{row reduce}}{\sim} [I \mid A^{-1}]$$

Elementary matrices:

$E_i \to$ performs a row reduction operation

$$E_i = \begin{bmatrix} 1 & & 0 \\ & \alpha & \\ 0 & & 1 \end{bmatrix} \qquad E_i = \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix} \qquad E_i = \begin{bmatrix} 1 & & & \\ & 1 & \alpha & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

scales a row by α — swaps two rows — adds a multiple of α times another row to a row

$$\to E_1[A \mid I] = [E_1A \mid E_1]$$

$$\to [E_2E_1A \mid E_2E_1] \qquad E_k \cdots E_1 = A^{-1}$$

$\vdots$

$$[E_k \cdots E_1 A = I \mid E_k \cdots E_1] \qquad A^{-1}A = I$$

$$y = Ax \qquad A^{-1}y = A^{-1}Ax = Ix = x$$

$$E_i = \begin{bmatrix} 1 & & 0 \\ & \alpha & \\ 0 & & 1 \end{bmatrix} \qquad E_i = \begin{bmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{bmatrix} \qquad E_i = \begin{bmatrix} 1 & & & \\ & 1 & \alpha & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \cdots \\ \alpha & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} — \bar{A}_1^T — \\ \vdots \\ — \bar{A}_m^T — \end{bmatrix} = \begin{bmatrix} — \bar{A}_1^T — \\ — \bar{A}_2^T — \\ — \alpha \bar{A}_i^T — \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} — \bar{A}_1^T — \\ — \bar{A}_2^T — \\ — \bar{A}_3^T — \end{bmatrix} = \begin{bmatrix} — \bar{A}_1^T — \\ \alpha \bar{A}_2^T — \\ — \bar{A}_3^T — \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} — \bar{A}_1^T — \\ — \bar{A}_2^T — \\ — \bar{A}_3^T — \end{bmatrix} = \begin{bmatrix} — \bar{A}_1^T — \\ \bar{A}_2^T + \alpha \bar{A}_3^T \\ — \bar{A}_3^T — \end{bmatrix}$$