→ 510 – Linear Systems

511 – Classical Control → · transfer functions · Steve Brunton

512 – Vehicle Dynamics / Control →]

→ 513 – Multivariable Control [- LQR, Lyapunov]

→ 514 – Estimation (Kalman Filters) ←

extended Kalman filter

unscented " "

particle "

OPTIMAL CONTROL ←

ROBUST CONTROL] ←

Steve Brunton – controls lecture series

data driven control

Brian Douglas – youtube

Mathworks

(Robust control) → Robust control toolbox

Outline

- more transfer functions
- Z-transform / Circulant
- SVD, Polar Decomp. – connections between $\mathbb{R}^{n\times n}$ & $\mathbb{C}$]

State space $\longrightarrow$ Transfer Func.

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{o\times n}$

$$\dot{x} = Ax + Bu \quad \Longrightarrow \quad G(s) = C(sI-A)^{-1}B + D \;\Leftarrow$$

$$y = Cx + Du$$

$$u(s) \longrightarrow \boxed{G(s)} \longrightarrow y(s)$$

Output:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Cx(t) + Du(t)$$

$$y(s) = C(sI-A)^{-1}x(0) + \left[C(sI-A)^{-1}B + D\right]u(s)$$

For a single input - single output SISO sys

$$G(s) \in \mathbb{C}$$

for multiple input - multiple output MIMO sys

$$G(s) \in \mathbb{C}^{o\times m} \qquad G(s) = \begin{bmatrix} G_{11}(s) & & G_{1m}(s) \\ & & \\ G_{o1}(s) & & G_{om}(s) \end{bmatrix}$$

SISO system:

$$G(s) = \frac{N(s)}{D(s)} = \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0}{\underbrace{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}_{\text{characteristic polynomial of } A}}$$

transfer func: $\longrightarrow$ state space (multiple options)

if you represent a state space model in a different set of coords still get the same transfer function

multiple coord systems...

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\rightarrow$ new coords $\underline{x = Px'}$

$$P\dot{x}' = APx' + Bu$$

$$\left.\begin{aligned} \dot{x}' &= P^{-1}APx' + P^{-1}Bu \\ y &= CPx' + Du \end{aligned}\right\} \rightarrow$$

transfer func...

$$G(s) = CP(sI - P^{-1}AP)^{-1}P^{-1}B + D$$
$$= C(sI-A)^{-1}B + D$$

one option:
put system in controllable canonical form..

$$\underline{G(s)} = \frac{N(s)}{D(s)} = \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0}{\underbrace{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}_{\chi_A(s) = \det(sI - \boxed{A})}}$$

find
$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \leftarrow$$ controllable canonical form

$$A = \begin{array}{c} \\ n \end{array}\begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -\alpha_0 & \cdots & & -\alpha_{n-1} \end{bmatrix} \qquad B = n\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = 1\,[\beta_0\ \beta_1\ \cdots\ \beta_{n-1}]$$

this state space model has transfer function $G(s)$

can check that

$$C(sI-A)^{-1}B = \underbrace{\frac{1}{\det(sI-A)}}_{D(s)} \underbrace{C\,\mathrm{Adj}(sI-A)B}_{N(s)} = \frac{\boxed{N(s)}}{D(s)}$$

Adjugate

$D(s)$: polynom. of s of deg n

$N(s)$: polynom of s of deg $n-1$

this state space had n states

$$u(s) \longrightarrow \boxed{G(s)} \longrightarrow y(s)$$

gears flywheels in box.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = \frac{N(s)}{D(s)} = \frac{\beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0}{s^{\boxed{n}} + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

could come up w/ A, B, C s.t. $A \in \mathbb{R}^{n' \times n'}$ where $n' > n$

$$G(s) = C(sI-A)^{-1}B = \frac{N(s)}{D(s)} = \frac{(s-z_1)\cancel{(s-z_i)}(s-z_{n-1})}{(s-\lambda_1)\cancel{(s-\lambda_j)}(s-\lambda_{n'})}$$

if $z_i = \lambda_j$

could still have deg. instead of n'

if you did this you've introduced uncontrollable or unobservable states into

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

if poles & zeros cancel in the transfer func

is unobservable or uncontrollable

if you pick A, B, C, D s.t. no poles or zeros cancel $\Rightarrow$ minimal realization

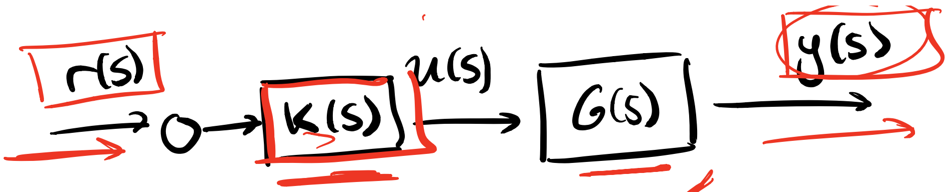
state space model w/ the fewest # of states that has transfer func. $G(s)$

Note:

- Never want to cancel unstable poles

$$\frac{\cancel{(s-z_i)}}{\cancel{(s-\lambda_j)}} \quad Re(\lambda_j) > 0$$

$\lambda_j = z_i$



$$y(s) = G(s)K(s)r(s)$$

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$$y(s) = \frac{\cancel{N(s)}}{\boxed{D(s)}}\left(\frac{\cancel{D(s)}}{\cancel{N(s)}}\right) r(s)$$

$$K(s) = G^{-1}(s):$$

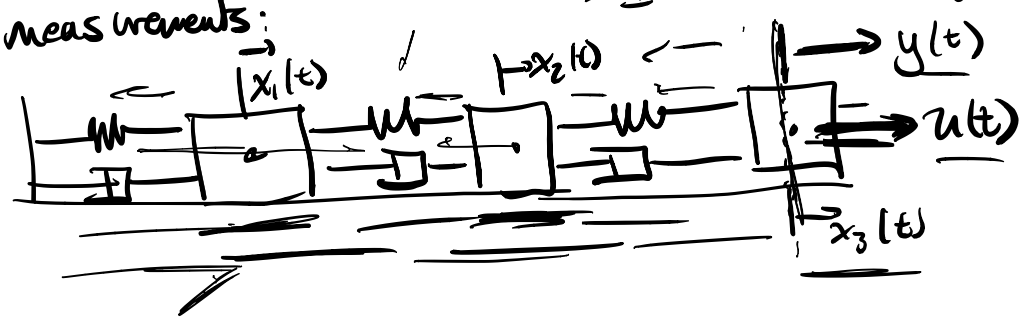
is this a good idea?

→ for unstable: $G(s)$ → no

→ for stable $G(s)$

→ I'm not sure.

transfer functions from measurements:



$$x \in \mathbb{R}^6 \quad \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & & & & \\ & & 0 & 1 & & \\ & & & & 0 & 1 \\ & & & & & \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = [0\ 0\ 0\ 0\ 1\ 0]\, x$$

$$y(t) \in \mathbb{R} \leftarrow \boxed{G} \leftarrow u(t) \in \mathbb{R}$$

$$\left[\, y(t) = a\,u(t) \,\right]$$

$$\frac{d^n y}{dt^n} + \alpha_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + \alpha_1 \frac{dy}{dt} + \alpha_0 = \beta_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + \cdots - \beta_1 \frac{du}{dt} + \beta_0$$

fit parameters of this model α_i β_j

↓

Laplace transform

$$s^n y(s) + \alpha_{n-1}s^{n-1}y(s) + \cdots \alpha_1 s\, y(s) + \alpha_0 y(s) = \beta_{n-1}s^{n-1}u(s) + \cdots + \beta_0 u(s)$$

$$+ \underbrace{H_y\left(y(0), \left.\frac{dy}{dt}\right|_0, \left.\frac{d^2y}{dt^2}\right|_0, \cdots\right)}_{\text{assume } = 0} \quad + \underbrace{H_u\left(u(0), \left.\frac{du}{dt}\right|_0, \cdots\right)}_{=0}$$

→ system is initially not moving

$$(s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0)\,y(s) = (\beta_{n-1}s^{n-1} + \cdots + \beta_0)\,u(s)$$

$$y(s) = G(s)\,u(s)$$

$$\left. G(s) = \frac{y(s)}{u(s)} = \frac{(\beta_{n-1}s^{n-1} + \cdots + \beta_0)}{(s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0)} \right]$$

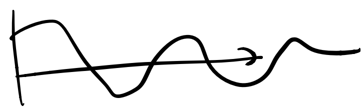
Z-tranforms & Circulant Matrices

Laplace transform: $\left[F(s) = \int_0^\infty f(t) e^{-st} dt \right]$

signal of a particular frequency s.

Discrete time:

$f(t)$



$\bar{f} = [\bar{f}(0), \bar{f}(1), \ldots, \bar{f}(k)]$

representing time signals as long vectors..

Discrete time Laplace-Transform.

Z-transform

Δt: time step.

k : # of time steps.

$t = \Delta t k$: total time

$$\left[\bar{F}(z) = \sum_{k=0}^{\infty} \bar{f}(k) z^{-k} \right]$$

$z \in \mathbb{C}$ $\quad z = e^{s\Delta t}$ $\quad z^{-k} = e^{-s\Delta t k}$

[s : frequency variable $\quad t = \Delta t k$

[$z = e^{s\Delta t}$: 1 step of evolution of a signal at frequency s

Circulant Matrices

$c \in \mathbb{R}^n \qquad c = \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix}$

$C \in \mathbb{R}^{n \times n}$

Circulant matrix for vector c.

discrete time representation of periodic signals

$$C = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & & c_1 \\ \vdots & c_0 & c_{n-1} & & \vdots \\ \vdots & \vdots & c_0 & & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & & c_0 \end{bmatrix}$$

shift matrices $\subset$ circulant matrices $\subset$ Toeplitz Matrices

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$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & \vdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & \vdots & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

S is shift matrix

represents taking a step in time

$$Sc = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & \vdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & \vdots & & 1 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} c_{n-1} \\ c_0 \\ \vdots \\ c_{n-2} \end{bmatrix}$$

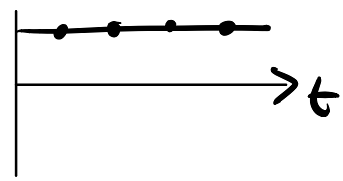
$$c = [\, c_0 \; c_1 \; \cdots \; c_{n-1} \,]$$

$$[\quad \underbrace{c_{n-1}\; c_0 \; c_1 \cdots \; c_{n-1}}\; c_0 \; c_1 \; \cdots$$

What are eigenvectors of the shift matrix S?

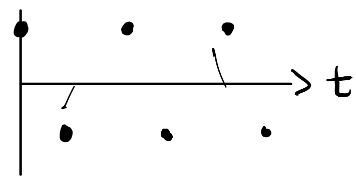
" " " "of stepping forward in time"

$$S\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}=\begin{bmatrix}0&0&\cdots&0&1\\ 1&0&&&0\\ 0&1&&&\vdots\\ \vdots&&\ddots&0&\vdots\\ 0&\cdots&&1&0\end{bmatrix}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$



S even dim. ex. $S\in\mathbb{R}^{4\times 4}$

$$\begin{bmatrix}0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\end{bmatrix}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ -1\\ 1\end{bmatrix}=(-1)\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix}$$



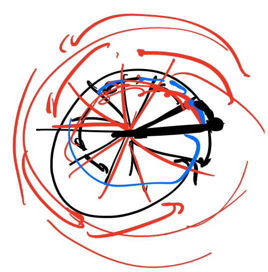
$$S\begin{bmatrix}1\\ e^{i\frac{k}{n}2\pi}\\ e^{i\frac{2k}{n}2\pi}\\ \vdots\\ e^{i\frac{(n-1)k}{n}2\pi}\end{bmatrix}=\begin{bmatrix}e^{i\frac{(n-1)k}{n}2\pi}\\ 1\\ e^{i\frac{k}{n}2\pi}\\ e^{i\frac{2k}{n}2\pi}\\ \vdots\end{bmatrix}=\begin{bmatrix}1\\ e^{i\frac{k}{n}2\pi}\\ e^{i\frac{2k}{n}2\pi}\\ \vdots\\ e^{i\frac{(n-1)k}{n}2\pi}\end{bmatrix}e^{-i\frac{k}{n}2\pi}=\begin{bmatrix}1\ e^{-i\frac{k}{n}2\pi}\\ e^{i\frac{k}{n}2\pi}\ e^{-i\frac{k}{n}2\pi}\\ e^{i\frac{2k}{n}2\pi}\ e^{-i\frac{k}{n}2\pi}\\ \vdots\\ e^{i\frac{(n-1)k}{n}2\pi}\ e^{-i\frac{k}{n}2\pi}\end{bmatrix}=\begin{bmatrix}e^{i\frac{(n-1)k}{n}}\\ 1\\ e^{i\frac{k}{n}}\\ e^{i\frac{2k}{n}}\\ \vdots\\ e^{i\frac{(n-2)k}{n}}\end{bmatrix}$$

const. slower faster

eigen values : $\underbrace{1=e^{-i\frac{0}{n}2\pi},\ e^{-i\frac{1}{n}2\pi},\ e^{-i\frac{2}{n}2\pi},\ \ldots,\ e^{-i\frac{(n-1)}{n}2\pi}}$ ←

$1=(e^{-i2\pi})$ $e^{-i2\pi}$

nth roots of unity ← $\left\{\ e^{-i\frac{k}{n}2\pi}\qquad k=0,\ldots,n-1\right.$



k : frequency of oscillation

taking steps around the unit circle at diff. speeds

$S \in \mathbb{R}^{2\times 2}$ $\quad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$C = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & & c_1 \\ c_1 & c_0 & c_{n-1} & & \vdots \\ \vdots & c_1 & c_0 & & \vdots \\ \vdots & \vdots & c_1 & & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & & c_0 \end{bmatrix} \longleftarrow c = [c_0\, c_1 \cdots c_{n-1}]$$

$$C = c_0 S^0 + c_1 S + c_2 S^2 + \cdots c_{n-1} S^{n-1}$$

C: circ. periodic signal

$S^0 = I$: no change

S: shift by 1

S^2: shift by 2

S^{n-1}: shift by $n-1$

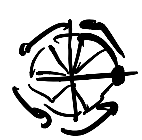
$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \\ 1 & 0 & & \vdots & 0 \\ 0 & 1 & & & \vdots \\ 0 & \vdots & & 0 & 0 \end{bmatrix} = S^2$$

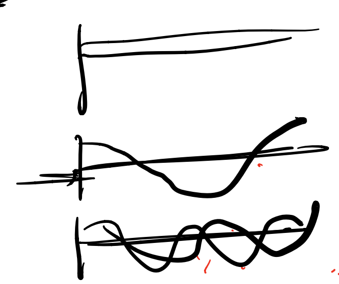
$$S^n = I \qquad S^{n+1} = S, \quad S^{n+2} = S^2 \ldots$$

Eigenvectors:

$$P = \begin{array}{c} \\ \text{eigenvectors} \end{array} \begin{array}{ccccc} k=0 & k=1 & k=2 & & k=n-1 \\ \left[\begin{matrix} 1 \\ \vdots \\ \vdots \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ e^{i\frac{2\pi}{n}} \\ e^{i2(\frac{2\pi}{n})} \\ e^{i(n-1)(\frac{2\pi}{n})} \end{matrix} & \begin{matrix} 1 \\ e^{i\frac{2\pi(2)}{n}} \\ e^{i\frac{2\pi 4}{n}} \\ e^{i2(n-1)\frac{2\pi}{n}} \end{matrix} & \cdots & \left.\begin{matrix} 1 \\ e^{i(n-1)\frac{2\pi}{n}} \\ \vdots \\ e^{i(n-1)(n-1)\frac{2\pi}{n}} \end{matrix}\right] \end{array}$$

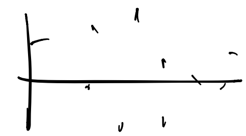
close to unitary





eigenvalues $1 \quad e^{-i\frac{2\pi}{n}} \quad e^{-i2(\frac{2\pi}{n})} \quad \cdots \quad e^{-i(n-1)\frac{2\pi}{n}}$

$$D = \begin{bmatrix} 1 & & & \\ & e^{-i\frac{2\pi}{n}} & & \\ & & \ddots & \\ & & & e^{-i(n-1)\frac{2\pi}{n}} \end{bmatrix}$$



$$S = PDP^{-1} \qquad P^{-1} = \frac{1}{n}P^*$$

$$= PDP^*\frac{1}{n}$$

easily diagonalize

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots + c_{n-1} S^{n-1}$$

$$= c_0 PP^*\frac{1}{n} + c_1 PDP^*\frac{1}{n} + c_2 PD^2P^*\frac{1}{n} + \cdots c_n PD^{n-1}P^*\frac{1}{n}$$

$$= P\Big[\underbrace{c_0 I + c_1 D + c_2 D^2 + \cdots + c_{n-1} D^{n-1}}_{\mathrm{diag}(P^* c)}\Big] P^*\frac{1}{n}$$

$\mathrm{diag}(\underbrace{P^* c}_{\mathrm{vec}}) \leftarrow$ some algebra.

$$I, D, D^2 = \begin{bmatrix} 1 & & & \\ & e^{-i\frac{2\pi}{n}2} & & \\ & & e^{-i\frac{2\pi}{n}4} & \\ & & & \ddots \end{bmatrix}, \ldots D^{n-1} = \begin{bmatrix} 1 & & & \\ & e^{-i\frac{2\pi}{n}(n-1)} & & \\ & & \ddots & \\ & & & e^{-\frac{2\pi(n-1)(n-1)}{n}} \end{bmatrix}$$

$$C = P\,\mathrm{diag}(P^* c)\,P^*\frac{1}{n}$$

$P = [p_1 \cdots p_n]$

$$P^* = \begin{bmatrix} - p_1^* - \\ \vdots \\ - p_n^* - \end{bmatrix} \begin{bmatrix} c \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ \vdots & e^{i\frac{2\pi}{n}} & e^{i\frac{2\pi(2)}{n}} & \cdots & e^{i\frac{(n-1)2\pi}{n}} \\ \vdots & e^{i2(\frac{2\pi}{n})} & e^{i\frac{2\pi 4}{n}} & & \vdots \\ 1 & e^{i(n-1)(\frac{2\pi}{n})} & e^{i2(n-1)\frac{2\pi}{n}} & & e^{i(n-1)(n-1)\frac{2\pi}{n}} \end{bmatrix}$$

columns are oscillations at different rates

$\longrightarrow$ discrete Fourier basis vectors of dim n.

$\frac{1}{\sqrt{n}} P$: DFT matrix — Discrete Fourier Transform matrix

$P^{\dagger} c$: discrete Fourier transform of periodic signal c.

if you represent a periodic signal c w.r.t to a different basis, namely the Fourier basis (cols of P), then propagating forward in time is actually represented by those Fourier basis vectors getting "stretched" by certain amounts

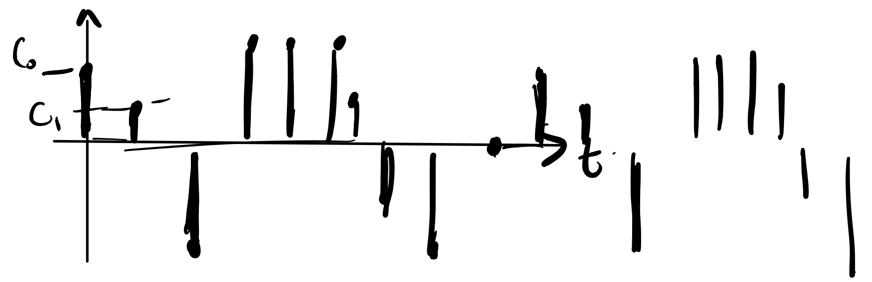
by complex # — discrete Fourier transform of the signal discrete frequency domain representation.

Big Picture:

$$c = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix}$$

$t=0 \quad t=1 \quad \cdots \quad t=n-1$

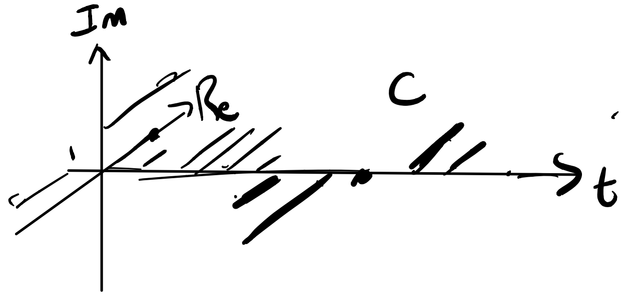
what coord sys. is "best" for representing time signals



operation: moving forward in time

$$c_0\ c_1\ \cdots\ c_{n-1}\ c_0\ c_1\ \cdots\ c_{n-1}$$

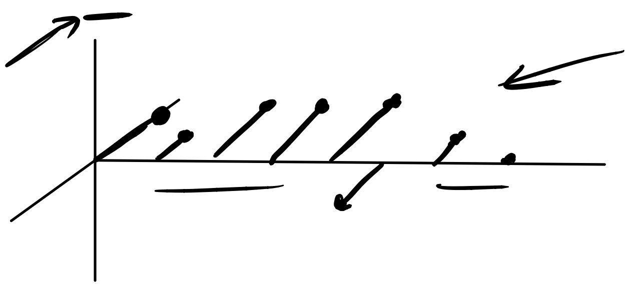
$$S := \begin{bmatrix} 0 & & 1 \\ 1 & & \\ \vdots & 1 & \\ 0 & & \ddots \end{bmatrix} \longrightarrow$$ oscillating signals are eigenvectors



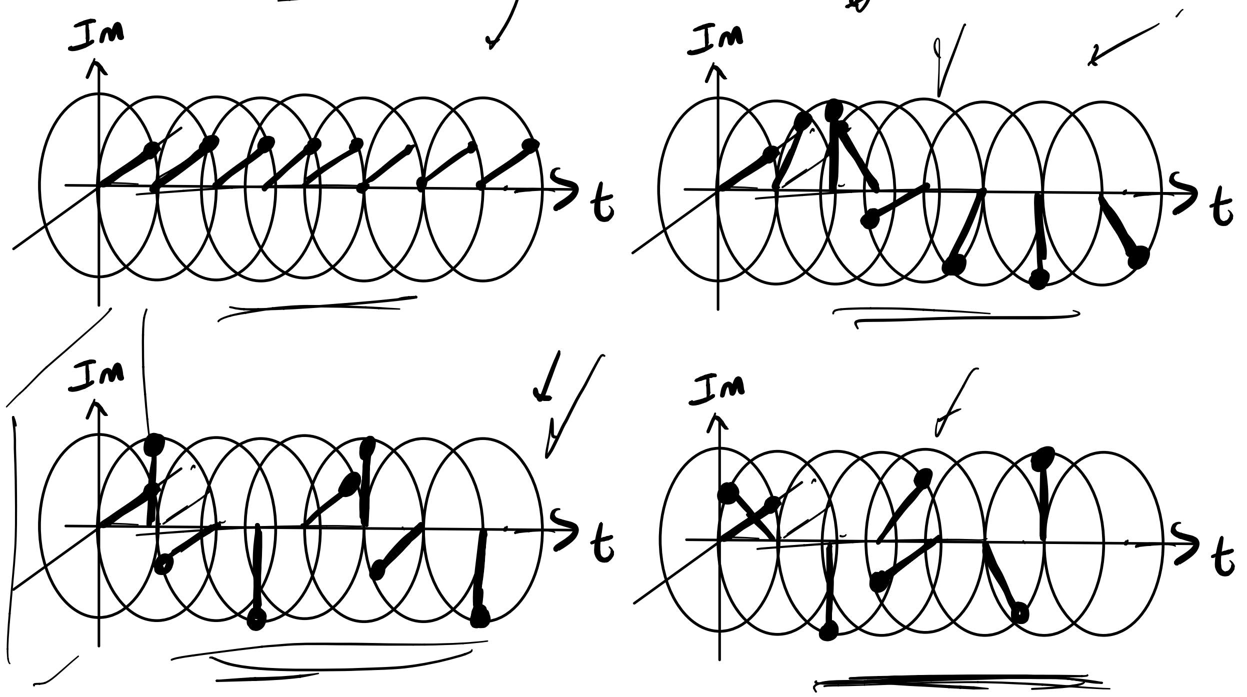
$c = [c_0 \; c_1 \; \cdots \; c_7] \leftarrow c'$: coords of c w.r.t. cols of P

$c = Pc'$

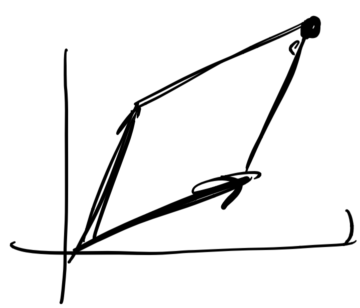
$\downarrow$ DFT of c



Columns of P: eigenvectors of $S \in \mathbb{R}^{8\times 8}$



$\vdots$



$$c = Pc' \longrightarrow c' = \frac{1}{n}P^* c$$

c': coords of c. w.r.t. eigenvectors of S

c': discrete fourier transform of c. (DFT)

$$C = P\,\mathrm{diag}(nc')\,P^*\frac{1}{n}$$

weird.

$$c = Pc'$$

$$Sc = PD\underbrace{P^{-1}P}_{I}c' = PDc'$$

$$\begin{bmatrix} P_0 \cdots P_{n-1} \end{bmatrix}\begin{bmatrix} 1 & & & \\ & e^{-\frac{i2\pi}{n}} & & \\ & & e^{-\frac{i2(2\pi)}{n}} & \\ & & & \ddots \end{bmatrix}\begin{bmatrix} c_0' \\ \vdots \\ c_{n-1}' \end{bmatrix}$$

$$Sc = P_0 c_0' + P_1 e^{-\frac{i2\pi}{n}} c_1' + P_2 e^{-\frac{i2\pi 2}{n}} c_2' + \cdots +$$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$y(t) = Ce^{At}\cancel{x(0)}^{0} + \int_0^t \underbrace{e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

Convolution of $u(t)$

w/ $e^{A(t)}$ control

sys reponse

$$y(t) = g(t) * u(t)$$

$$Y(s) \quad G(s) \quad U(s)$$

$$Y(s) = G(s)U(s)$$