Basis: set of vectors
- span a space
- lin ind.

$\mathbb{R}^n \quad x \in \mathbb{R}^n$

basis: cols of $P \in \mathbb{R}^{n \times n}$

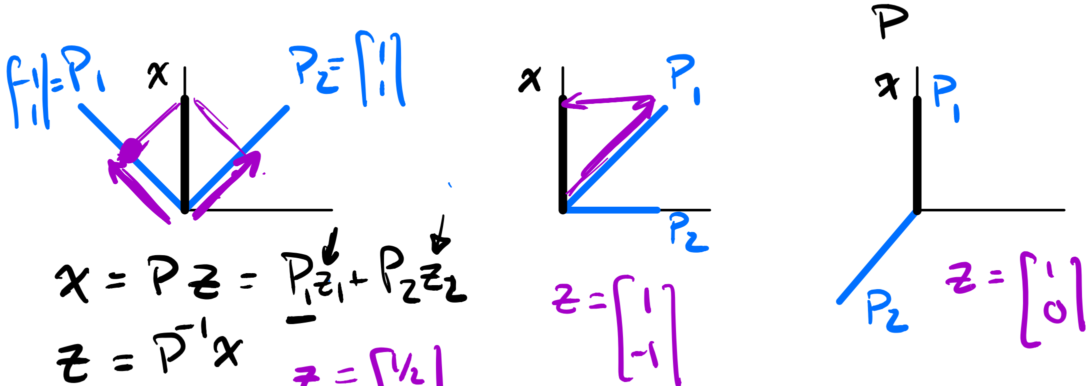
$$P = [p_1 \cdots p_n]$$

$$x = Pz = p_1 z_1 + \cdots + p_n z_n$$

basis vectors

coords of x w.r.t. the basis P

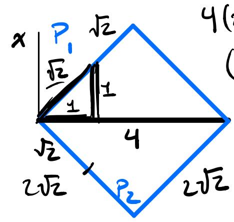
Ex: $P = [p_1\ p_2]$



$x = Pz = p_1 z_1 + p_2 z_2$

$z = P^{-1} x \qquad z = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$



$4(2) + 4(2) = 16$

$(2\sqrt{2})^2 + (2\sqrt{2})^2$

P is a basis for $\mathbb{R}^n$

changing basis = coordinate transformation

P: new basis, coordinate transform from the standard basis

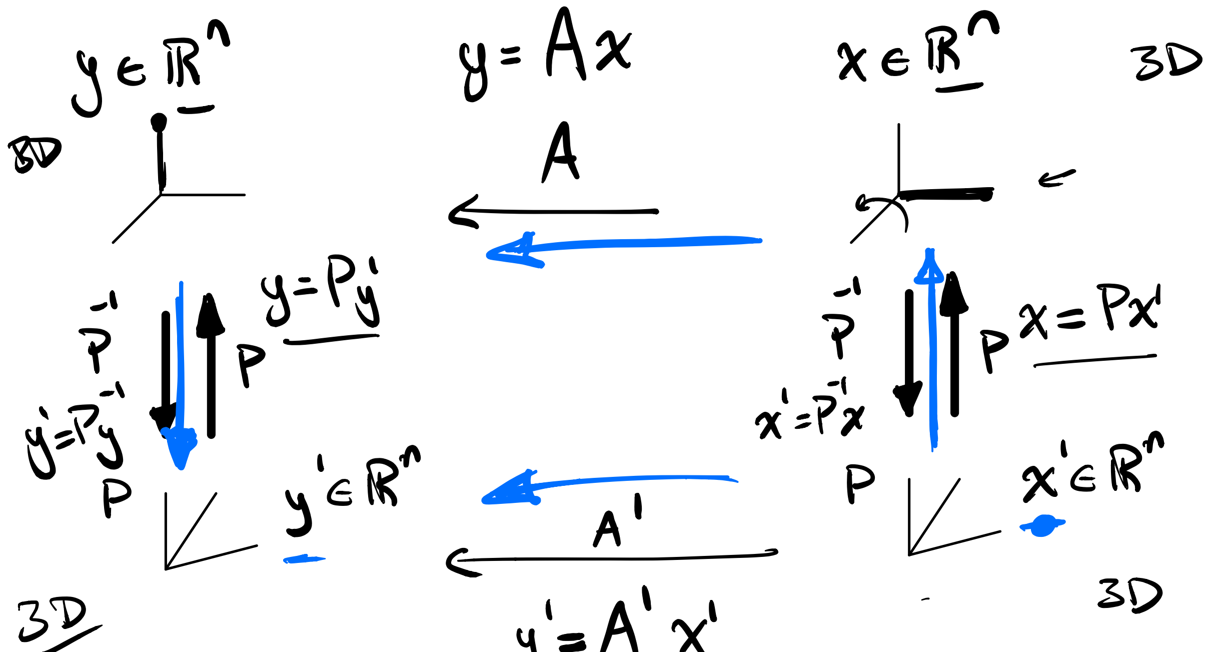
$$x = Px' \rightarrow \text{coords w.r.t. } P.$$

coords w.r.t standard basis

Inverse coord transform: $x' = P^{-1}x$

How do coord transforms affect matrices?

$A \in \mathbb{R}^{n \times n}$ square



$y' = A'x'$

$$y = Ax$$

$$Py' = APx'$$

$$y' = P^{-1}APx'$$

$$\boxed{A' = P^{-1}AP}$$

similarity transform on A

A & A' are similar

Two similar matrices perform the same transformation but w.r.t different coord. systems.

$$P(A' = P^{-1}AP)P^{-1} \implies \boxed{A = PA'P^{-1}}$$

Note : in general matrices don't commute

$AB \neq BA$ $\rightarrow$ [distributivity, associativity] [commutativity] More general

2 rotations... R_1, R_2

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{2D} \quad y \in \mathbb{R}^m \arrow[d, shift right, "y' = Q^{-1} y"'] & x \in \mathbb{R}^n \quad \text{3D} \arrow[l, "\substack{y = Ax \\ A \in \mathbb{R}^{m \times n}}"'] \arrow[d, shift right, "x' = P^{-1} x"'] \\
\text{2D} \quad y' \in \mathbb{R}^m \arrow[u, shift right, "y = Q y'"'] & x' \in \mathbb{R}^n \quad \text{3D} \arrow[l, "A' \in \mathbb{R}^{m \times n}"', "y' = A' x'"] \arrow[u, shift right, "x = P x'"']
\end{tikzcd}
$$

$$y = Ax$$

$$Qy' = APx' \Rightarrow y' = Q^{-1}APx' \quad A' = Q^{-1}AP$$

(Similarity) when $Q = P$.

$$y = Ax$$

$$
\begin{tikzcd}[column sep=huge]
y \in \mathbb{R}^m & x \in \mathbb{R}^n \arrow[l, "\substack{A \in \mathbb{R}^{m \times n} \\ A}"']
\end{tikzcd}
$$

CODOMAIN ($\mathbb{R}^m$) DOMAIN ($\mathbb{R}^n$)

1. Range(A) $= \mathcal{R}(A) \subseteq$ CODOMAIN.
2. Nullspace(A) $= \mathcal{N}(A) \subseteq$ DOMAIN

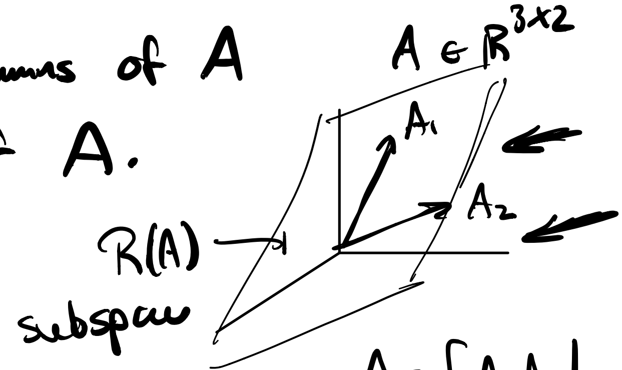
Range(A^T) $= \mathcal{R}(A^T) \subseteq$ DOMAIN

Nullspace(A^T) $= \mathcal{N}(A^T) \subseteq$ CODOMAIN

Range & Nullspace

1. <u>Range</u> $\mathcal{R}(A) = \{ y \in \mathbb{R}^m \mid y = Ax,\ x \in \mathbb{R}^n \}$

$\mathcal{R}(A)$ is span of columns of A
"columns space of A.



$A = [A_1, A_2]$

2. <u>Nullspace</u> $\mathcal{N}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0,\ x \in \mathbb{R}^n \}$

0 vector

if $\mathcal{R}(A) = \mathbb{R}^m$: A is <u>onto</u> (<u>surjective</u>)

$\exists\, x$ s.t. $y = Ax$ for any $y \in \mathbb{R}^m$

if $\mathcal{N}(A) = \{0\}$ $\rightarrow$ A has a trivial nullspace

$\times$ $\mathcal{N}(A) = 0$ A is <u>one-to-one</u> (in<u>jective</u>)

if $y = Ax$ then x is unique
only x that maps to y.

if not $y = Ax$, $y = Ax'$ for $\underline{x \neq x'}$ $Ax = y = Ax'$

$$\Rightarrow A(\underbrace{x - x'}_{\neq 0}) = 0$$

more than 1 solution to a system of lin. eqns. means a non trivial nullspace.

Some properties of $N(A)$:

- if $x \in N(A)$, then $x \perp$ rows of A.

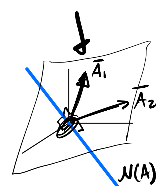
$$A = \begin{bmatrix} - \bar{A}_1^T - \\ \vdots \\ - \bar{A}_m^T - \end{bmatrix} \quad Ax = \begin{bmatrix} - \bar{A}_1^T - \\ \vdots \\ - \bar{A}_m^T - \end{bmatrix} x = \begin{bmatrix} \bar{A}_1^T x \\ \vdots \\ \bar{A}_m^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{A}_i^T x = 0 \quad i = 1, \dots, m$$

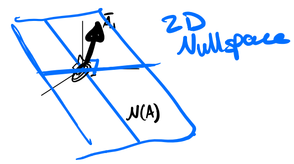
Picture

DOMAIN

$A \in \mathbb{R}^{2\times 3}$ $\quad A = \begin{bmatrix} -\bar{A}_1^T- \\ -\bar{A}_2^T- \end{bmatrix}$



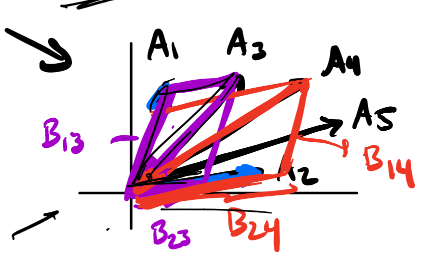
$A \in \mathbb{R}^{1\times 3}$ $\quad A = [-\bar{A}_1^T-]$



Construct a basis for $N(A)$:

$A \in \mathbb{R}^{2\times 5}$ $\quad A = [A_1\ A_2\ A_3\ A_4\ A_5]$

CODOMAIN



$y = Ax$ non unique solutions

cols of A are lin dep.

$z \in N(A)$

$$y = A(x + z) = Ax + \cancel{Az}^{\,0}$$

$$y = [\underline{A_1 \cdots A_n}]\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \right.$$

- $A = [\underbrace{A_1 A_2}_{\text{lin ind cols}} \mid \underbrace{A_3 A_4 A_5}_{\text{lin dep cols}}]$

$A_3 = [A_1 A_2] B_3 \rightarrow$ coords of A_3 w.r.t. A_1 & A_2 $\quad B_3 = \begin{bmatrix} B_{13} \\ B_{23} \end{bmatrix}$

$A_4 = [A_1 A_2] B_4$ — coords of A_4 & A_5 w.r.t. A_1 & A_2 $\quad B_4 = \begin{bmatrix} B_{14} \\ B_{24} \end{bmatrix}$

$A_5 = [A_1 A_2] B_5$

Consider:

$$[A_1 \; A_2 | A_3 \; A_4 \; A_5] \begin{bmatrix} B_3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = [A_1 A_2] B_3 - A_3 = 0$$

$$[A_1 A_2 | A_3 A_4 A_5] \begin{bmatrix} B_4 \\ 0 \\ -1 \\ 0 \end{bmatrix} = [A_1 A_2] B_4 - A_4 = 0$$

$$\begin{array}{c c} & 3 \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} B_3 & B_4 & B_5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{array} \rightarrow \text{cols are a basis for } N(A).$$

General: $A \in \mathbb{R}^{m \times n}$ w/ k lin ind cols

- $A = [\underbrace{A_1 \cdots A_k}_{\text{lin ind}} | \underbrace{A_{k+1} \cdots A_n}_{\text{lin dep on}}]$

- find coords of $A_{k+1} \cdots A_n$ wrt. $[A_1 \cdots A_k]$

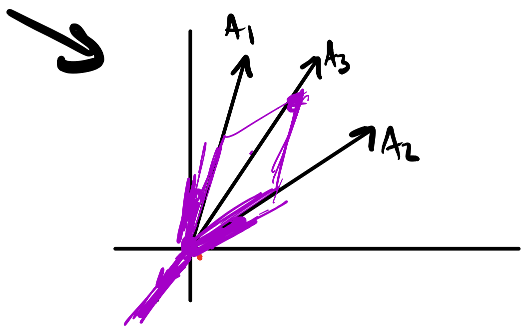
$A_j = [A_1 \cdots A_k] B_j \quad j = k+1, \ldots, n$

- $B = [B_{k+1} \cdots B_n]$

- $N = \begin{array}{c c} & n-k \\ \begin{array}{c} k \\ + \\ n-k \end{array} & \begin{bmatrix} B \\ -I \end{bmatrix} \end{array}$ $\quad N \in \mathbb{R}^{(k+(n-k)) \times (n-k)}$, $\quad N \in \mathbb{R}^{(n \times (n-k))}$

- $AN = 0$ → every column of N is in nullspace of A

$A \in \mathbb{R}^{2\times 3}$



$A \in \mathbb{R}^{m\times n}$ w k lin ind cols...
$\rightarrow$ find $n-k$ vectors in nullspace

cols of $N = \begin{bmatrix} B \\ -I \end{bmatrix}$

- cols are lin ind.
- $\exists z$ s.t. $x = Nz$ for any $x \in N(A)$

$\boxed{\text{lin ind:}}$

lindep $\exists z \neq 0$
$Az = 0$

$-\frac{A_1 z_1}{z_j} + \cdots + A_j z_j = 0$

Assume $Nz = 0$
show $z = 0$

$$Nz = \begin{bmatrix} Bz \\ -z \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow z = 0$$

$$N = \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{k+1} & \cdots & B_n \\ -1 & 0 \cdots & 0 \\ \vdots & & \\ 0 & \cdots 0 & -1 \end{bmatrix}$$

if $\underline{x \in N(A)}$ then $\exists$ z s.t. $x = Nz$ $\boxed{\text{span}}$

$$A = [A_1 \cdots A_k \mid A_{k+1} \cdots A_n]$$

$$[\underline{A_1 \cdots A_k} \mid [\underline{A_1 \cdots A_k}][B_{k+1} \cdots B_n]]$$

B

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Ax = [A_1 \cdots A_k][I \mid B]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\underline{A_1 \cdots A_k}][\underline{x_1 + Bx_2}] = 0$$

lin ind.

$$x_1 + Bx_2 = 0 \Rightarrow -Bx_2 = +x_1$$

$$\underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -Bx_2 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} B \\ -I \end{bmatrix}}_{\underline{N}}(-x_2) \qquad z = -x_2$$

$\boxed{\text{$A \in \mathbb{R}^{m \times n}$ w k lin ind cols... $\to$ find $n-k$ vectors in nullspace}}$

$$\# \text{ of lin ind cols} + \dim N(A) = n$$

$$\underbrace{k}_{\text{rank}(A)} + n - k = n$$

Rank Nullity Thm: $A \in \mathbb{R}^{m \times n}$

$$\boxed{\text{rk}(A) + \underbrace{\dim N(A)}_{\text{nullity}} = n}$$

Matrix rank:

col rank: # of lin ind cols.

row rank: # of lin ind rows.

col rank = row rank = rank

PROOF CLEVER. (END OF CLASS)

$$\boxed{rk(A) = rk(A^T) = rk(A^TA) = rk(AA^T)}$$

A^TA, AA^T: Grammians pseudo inverses

$A \in \mathbb{R}^{m\times n}$ [ ]

$A^TA \in \mathbb{R}^{n\times n}$ $AA^T \in \mathbb{R}^{m \times m}$

$$\boxed{\underset{\mathbb{R}^{m\times n}}{rk(A)} = rk(\underbrace{A^TA}_{\in \mathbb{R}^{n\times n}}):}$$

proof: $N(A) = N(A^TA)$ ← rank-nullity

Assume $Ax = 0 \implies A^T\underbrace{Ax}_{0} = 0$ ✓

$A^TAx = 0 \implies x^TA^TAx = 0$ $|Ax| = 0$

$|Ax|^2 = 0$ $Ax = 0$ ✓

A tall m[ ] $m > n$ A fat m[ ]

$A^TA = [\quad][\ ] = [\ ]$ $\qquad$ $A^TA = [\ ][\quad] = [\quad]$

$AA^T = [\ ][\quad] = [\quad]$ $\qquad$ $AA^T = [\quad][\ ] = [\ ]$

if A not square one of these Gramians is big & other is small.

if A has rank n: A^TA invertible

if A has rank m: AA^T invertible

A is full col rank if it's cols are lin ind $rk(A)=n$

A is full row rank if it's rows are lin ind $rk(A)=m$

square matrix A

- A is invertible / non singular
- A is full row & col rank $\leftarrow$

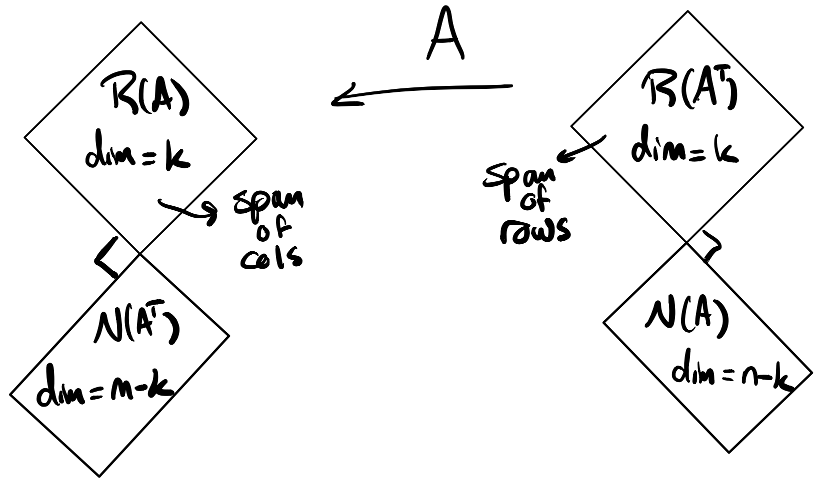
Fundamental Thm of Linear Algebra

$y = Ax$

$A \in \mathbb{R}^{m \times n}$

$rk(A) = k$

CODOMAIN ($\mathbb{R}^m$) DOMAIN ($\mathbb{R}^n$)



V & W are $\perp$ if $v \in V, w \in W \Rightarrow v^T w = 0$

$R(A) \perp N(A^T)$ $R(A^T) \perp N(A)$

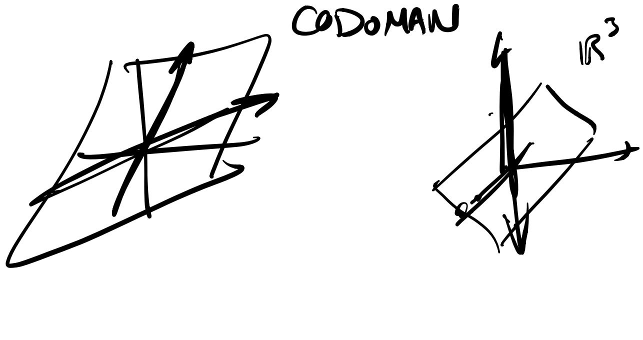
(vectors in $N(A)$ orthogonal to rows.)

vector space Direct Sum of 2 subspaces

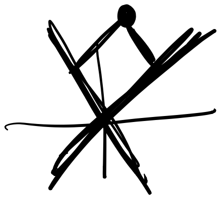
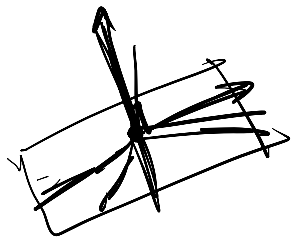
$V \oplus W = \{v + w \mid \forall v \in V, \forall w \in W\}$

$R(A) \oplus^{\perp} N(A^T) = \mathbb{R}^m$ $R(A^T) \oplus^{\perp} N(A) = \mathbb{R}^n$

CODOMAIN DOMAIN



$R(\{v, w\})$

Systems of Linear Equations

$y = Ax$

A tall

→ A full col rank

→ A^TA invertible

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

y might not be in range of A.

$\boxed{\text{no solution}}$

(overconstrained)

↓

closest x.

$$\min_x |y - Ax|^2$$

⇒ LEAST SQUARES Regression

A square

(invertible)

$\boxed{\text{a unique solution}}$

$$\boxed{x = A^{-1} y}$$

A fat.

A full row rank

(AA^T) invertible

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

$\boxed{\text{a continuum or subspace of solns}}$

if $y = Ax$

& $\text{span}(N) = \mathcal{N}(A)$

$x + Nz$ is also a soln for any z

$Ax + \underbrace{AN}_{0} z$

smallest x.

$$\min_x |x|^2$$

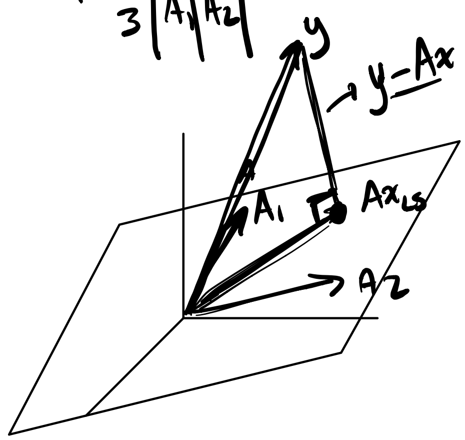
s.t. $y = Ax$

MIN. NORM SOLN.

LEAST SQUARES

$\rightarrow \min_x |y - Ax|^2 = (y-Ax)^T(y-Ax)$

ex. $A = {}_3\overset{2}{[A_1 | A_2]}$



$\min_x \; y^Ty - 2y^TAx + x^TA^TAx = J(x)$

$\frac{\partial J}{\partial x} = 0: \; -2y^TA + 2x^TA^TA = 0$

$x^T(A^TA) = y^TA$

$x^T = y^TA(A^TA)^{-1}$

$$\boxed{x_{LS} = (A^TA)^{-1}A^Ty}$$

$\text{proj}_A y = Ax_{LS} = A(A^TA)^{-1}A^Ty$

$y(t) = [A_{t1} \cdots A_{tn}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ **model**

$\begin{bmatrix} y_1 \\ \\ y_T \end{bmatrix} = \underset{\text{meas}}{\updownarrow} \overset{\text{params}}{\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ & \vdots & \\ A_{T1} & \cdots & A_{Tn} \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Big] \rightarrow$ param

MIN NORM:

$y = [-A-]\begin{bmatrix} x \end{bmatrix}$

$x = x_0 + Nz$

$\min_x |x|^2$

s.t. $y = Ax$

$$\boxed{x_{mn} = A^T(AA^T)^{-1}y}$$

$y = Ax_{mn}$

$= AA^T(AA^T)^{-1}y$

$= y$

arbitrary lin comb.

$\rightarrow x = x_{mn} + Nz \quad R(N) = N(A)$

$|x|^2 = x^Tx = x_{mn}^Tx_{mn}$

$+ 2x_{mn}^TNz$

$+ z^TN^TNz$

$= y^T(AA^T)^{-1}\underbrace{\cancel{(AA^T)^{-1}AA^T}}_{I}(AA^T)^{-1}y$

$+ 2y^T(AA^T)^{-1}\cancel{AN}z \quad 0$

$+ z^TN^TNz \quad |Nz|^2$

$(z^T N^T)(Nz)$ $\quad = y^T (AA^T)^{-1} y + |Nz|^2$

$\Rightarrow z = 0$

$(AA^T)^{-1}(AA^T)$

$|x|^2 = y^T (AA^T)^{-1} y$

$$\boxed{x_{LS} = (A^T A)^{-1} A^T y} \qquad \boxed{x_{mn} = A^T (AA^T)^{-1} y}$$

$(A^T A)^{-1} A^T$: left inverse

$(A^T A)^{-1} A^T A = I$ $\qquad A(A^T A)^{-1} A^T$

$(A^T (AA^T)^{-1})$: right inverse

$AA^T (AA^T)^{-1} = I$ $\qquad A^T (AA^T)^{-1} A$

$(A^T A)^{-1} A^T \qquad A^T (AA^T)^{-1}$

$A^T (A^T A)^{-1}$

Moore Penrose Pseudo inverse:

$x = A^+ y$ ← min norm least squares solution

Singular Value Decomposition