HW 2 PROBLEM 7 $\quad A \in \mathbb{R}^{m\times n}$

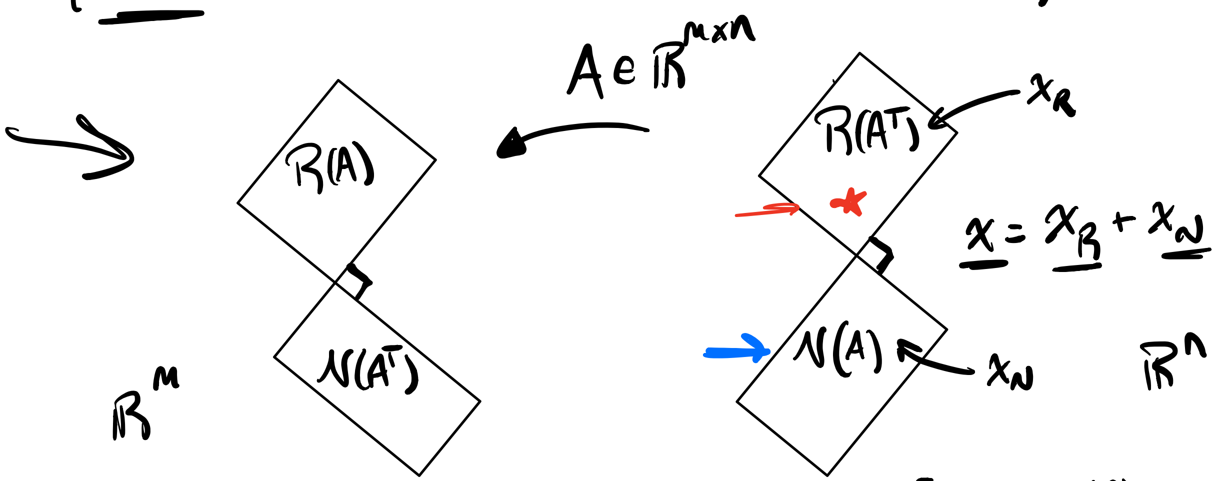
$$n\overset{\quad m \;\; +(n-m)}{\begin{bmatrix} A^T & N \end{bmatrix}} \in \mathbb{R}^{n\times n}$$

N : cols basis for $\mathcal{N}(A)$

A : lin ind rows

$N \in \mathbb{R}^{n\times(n-m)} \longrightarrow$ # of lin ind vectors in $\mathcal{N}(A)$

$$\underline{\begin{bmatrix} A^T & N \end{bmatrix}}^{-1} = ?$$



CODOMAIN $= \mathcal{R}(A) \oplus \mathcal{N}(A^T)$ $\qquad$ $\underline{\text{DOMAIN}} = \underline{\mathcal{R}(A^T)} \oplus \underline{\mathcal{N}(A)}$

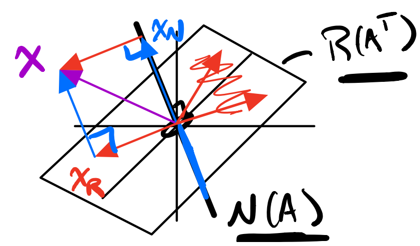
$x \in \mathbb{R}^n$

Coord transform $\Longrightarrow$ $x = \underline{\begin{bmatrix} A^T & N \end{bmatrix}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $\quad z_1 \in \mathbb{R}^m,\; z_2 \in \mathbb{R}^{n-m}$

$$= \underbrace{\underline{A^T z_1}}_{x_R} + \underbrace{\underline{N z_2}}_{x_N}$$

$$x = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\underline{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}} = \begin{bmatrix} A^T & N \end{bmatrix}^{-1} x$$



$$\begin{bmatrix} A^T & N \end{bmatrix}^{-1} \overset{?}{=} \begin{bmatrix} A \\ N^T \end{bmatrix} = \begin{bmatrix} A^T & N \end{bmatrix}^T$$

$$\underline{\begin{bmatrix} A \\ N^T \end{bmatrix}} \underline{\begin{bmatrix} A^T & N \end{bmatrix}} = \begin{bmatrix} AA^T & \cancel{AN}^{\,0 \in \mathbb{R}^{m \times n-m}} \\ \cancel{N^T A^T}_{\,0} & N^T N \end{bmatrix} = \underline{\begin{bmatrix} AA^T & 0 \\ 0 & N^T N \end{bmatrix}}$$

$$A\,N = A \begin{bmatrix} N_1 & \cdots & N_{n-m} \end{bmatrix}$$

$$\begin{bmatrix} AA^T & 0 \\ 0 & N^T N \end{bmatrix}^{-1} = \begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^T N)^{-1} \end{bmatrix} \qquad = \begin{bmatrix} \underbrace{AN_1}_{0} & \cdots & \underbrace{AN_{n-m}}_{0} \end{bmatrix}$$

$$\underbrace{\underline{\begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^T N)^{-1} \end{bmatrix} \begin{bmatrix} A \\ N^T \end{bmatrix}}}_{\begin{bmatrix} A^T & N \end{bmatrix}^{-1}} \underline{\begin{bmatrix} A^T & N \end{bmatrix}} = \underline{\begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^T N)^{-1} \end{bmatrix}} \underline{\begin{bmatrix} AA^T & 0 \\ 0 & N^T N \end{bmatrix}}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} (AA^T)^{-1} A \\ (N^T N)^{-1} N^T \end{bmatrix} \qquad x = \begin{bmatrix} A^T & N \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (AA^T)^{-1} A x \\ (N^T N)^{-1} N^T x \end{bmatrix} \begin{matrix} \Leftarrow \\ \Leftarrow \end{matrix}$$

$\mathrm{proj}_{A^T} x$ $\qquad$ $\mathrm{proj}_N x$

$$x = A^T z_1 + N z_2 = \underbrace{A^T (AA^T)^{-1} A x}_{x_R} + \underbrace{N (N^T N)^{-1} N^T x}_{x_N}$$

$$\boxed{\text{Row } \underset{\substack{\downarrow \\ \#\text{ of lin ind} \\ \text{rows}}}{\underline{\text{rank}}} = \text{col } \underset{\substack{\downarrow \\ \#\text{ of lin ind} \\ \text{cols}}}{\underline{\text{rank}}}} \Leftarrow \quad \Leftarrow$$

$\leftarrow$ unknowns $\rightarrow$

n

$A \in \mathbb{R}^{m \times n}$ eqns $\updownarrow m$

k

Sys eqns: n equations

n unknowns

WTS: $\left.\begin{array}{l} r \le k \\ k \le r \end{array}\right\} \rightarrow r = k$

row rank : r

$\rightarrow$ col rank : $\underline{k}$

1) $\boxed{r \le k}$

lin ind & form basis for $\mathcal{R}(A)$

$$C = \begin{bmatrix} c_1 \cdots c_k \end{bmatrix} = \begin{bmatrix} - \bar{c}_1^T - \\ \vdots \\ - \bar{c}_m^T - \end{bmatrix} \in \mathbb{R}^{m \times k}$$

$$A = \underline{CV}$$

$$V = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} = \begin{bmatrix} - \bar{v}_1^T - \\ \vdots \\ - \bar{v}_k^T - \end{bmatrix} \in \mathbb{R}^{k \times n}$$

$$= \begin{array}{c} \\ m \end{array} \overset{k}{\begin{bmatrix} c_1 \cdots c_k \end{bmatrix}} \begin{array}{c} \\ k \end{array} \overset{n}{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}} = \begin{bmatrix} Cv_1 \; Cv_2 \cdots Cv_n \end{bmatrix}$$

cols are basis for $\mathcal{R}(A)$

coeffs of ea. col of A w.r.t. the C basis

$$= \underbrace{\begin{bmatrix} - \bar{c}_1^T - \\ \vdots \\ - \bar{c}_m^T - \end{bmatrix}}_{\text{coeffs of rows of } A} \left.\begin{bmatrix} - \bar{v}_1^T - \\ \vdots \\ - \bar{v}_k^T - \end{bmatrix}\right\} \rightarrow \text{spanning } \mathcal{R}(A^T)$$

$$\begin{bmatrix} A_1 \cdots A_k \end{bmatrix} \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & v_{k+1} \cdots v_n \\ 0 & & 1 & \end{array}\right]$$

$$A_{k+1} = \begin{bmatrix} A_1 \cdots A_k \end{bmatrix} v_{k+1}$$

$$= \begin{bmatrix} - \bar{c}_1^T V - \\ - \bar{c}_m^T V - \end{bmatrix} \leftarrow \text{lin comb of rows of } V$$

$\Rightarrow$ ea row of A is a lin comb of rows of V

V has $\underline{k}$ $\underline{\text{rows}}$

$$\Rightarrow \dim \mathcal{R}(A^T) \leq k \Rightarrow \boxed{r \leq k}$$

2) $\boxed{k \leq r}$

$$A = \underbrace{W}_{} \underbrace{R}_{}$$

ea row is coeffs of that row of A w.r.t. basis of rows of R

rows are basis for $\mathcal{R}(A^T)$

$$R \in \mathbb{R}^{r \times n}$$
$$W \in \mathbb{R}^{m \times r}$$

$$R = \begin{bmatrix} - \bar{R}_1^T - \\ \vdots \\ - \bar{R}_r^T - \end{bmatrix} = \begin{bmatrix} R_1 \cdots R_n \end{bmatrix}$$

$$W = \begin{bmatrix} - \bar{W}_1^T - \\ - \bar{W}_m^T - \end{bmatrix} = \begin{bmatrix} W_1 \cdots W_r \end{bmatrix}$$

span.pdf.

wikipedia (rank)

$$= \begin{bmatrix} - \bar{W}_1^T - \\ - \bar{W}_m^T - \end{bmatrix} \begin{bmatrix} - \bar{R}_1^T - \\ \vdots \\ - \bar{R}_r^T - \end{bmatrix} \leftarrow \text{rows are basis}$$

rows are coeffs

$$= \begin{bmatrix} W_1 \cdots W_r \end{bmatrix} \begin{bmatrix} R_1 \cdots R_n \end{bmatrix}$$

span of cols of A $\leq r$ $\Rightarrow \boxed{k \leq r}$

Inverses

systems of eqns: $y = Ax$

$A \in \mathbb{R}^{n \times n}$ invertible $\rightarrow$ row reduce $A \sim I$

Gaussian Elimination / Row reduction

solve for x $\quad y = Ax \Leftarrow \quad \not\Rightarrow A^{-1}y = x$

Augmented system: $\quad [A \mid y] \underset{}{\overset{\text{row operations}}{\sim}} [I \mid x]$

Elementary Matrices E_i

$$\underbrace{E_i [A \mid y]}_{\text{performs a row operation}} = [E_i A \mid E_i y]$$

$$E_k \cdots E_2 E_1 [A \mid y] = [\underbrace{E_k \cdots E_2 E_1 A}_{\substack{A^{-1}A \\ I}} \mid \underbrace{E_k \cdots E_2 E_1 y}_{A^{-1}y = x}]$$

$$\boxed{A^{-1} = E_k \cdots E_2 E_1} \leftarrow$$

Elementary Matrices perform row operations:

Ex.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - A_1^T - \\ - A_2^T - \\ - A_3^T - \end{bmatrix} = \begin{bmatrix} - A_2^T - \\ - A_1^T - \\ - A_3^T - \end{bmatrix}$$

switching rows

$$\begin{array}{c} \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{array} \begin{bmatrix} I & 0 & 0 \\ 0 & & 1 \\ 0 & I & 0 \\ 1 & & 0 \\ 0 & 0 & I \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - A_1^T - \\ - A_2^T - \\ - A_3^T - \end{bmatrix} = \begin{bmatrix} - A_1^T - \\ \alpha A_2^T - \\ - A_3^T - \end{bmatrix}$$

scaling a row

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} - \bar{A}_1^T - \\ - \bar{A}_2^T - \\ - \bar{A}_3^T - \end{bmatrix} = \begin{bmatrix} \alpha_1 \bar{A}_1^T + \alpha_2 \bar{A}_2^T \\ - \bar{A}_2^T - \\ - \bar{A}_3^T - \end{bmatrix}$$

$$\begin{bmatrix} I & \alpha_i \cdots \alpha_j & \\ & I & \\ & & I \end{bmatrix}$$

lin combs. of rows

Equiv: row reduce $[A \mid I] \sim [I \mid A^{-1}]$

A not square: $A \in \mathbb{R}^{m \times n}$

fat: $A = [A^1 \; A^2]$ $\quad A^1 \in \mathbb{R}^{m \times m}$

$m < n$ $\quad A^2 \in \mathbb{R}^{m \times n-m}$

$$[A^1 A^2 \mid y] \sim [\boxed{I} \; \boxed{(A^1)^{-1} A^2} \mid (A^1)^{-1} y]$$

$$y = [A^1 A^2] \begin{bmatrix} (A^1)^{-1} y \\ 0 \end{bmatrix} = (A^1)(A^1)^{-1} y + 0$$

x

$$x = \begin{bmatrix} (A^1)^{-1} y \\ 0 \end{bmatrix} + \begin{bmatrix} -(A^1)^{-1} A^2 \\ I \end{bmatrix} z$$

specific soln

$$A \sim [\;] \sim I$$

$$I [A^1 A^2] = A^1 (A^1)^{-1} [A^1 \; A^2]$$

$$= A^1 [I \; (A^1)^{-1} A^2]$$

if matrix doesn't have m lin ind rows

basis cols

coeffs A^2 w.r.t. A^1

$$\begin{bmatrix} -(A^1)^{-1}A^2 \\ I \end{bmatrix}$$

$$A \sim \left[\begin{array}{c|ccc} I & & & \\ 0 & 0 & \cdots & 0 \end{array}\right]$$

A tall full col rank

$A^{-1}A = I \quad A^{-1}y = x$

$$A = \left[\begin{array}{c|c} \bar{A}^1 & y_1 \\ \bar{A}^2 & y_2 \end{array}\right] \sim \left[\begin{array}{c|c} I & * \\ \hline 0 & \boxed{\xi} \end{array}\right]$$

$$= [A \mid y] \sim \left[\begin{array}{cc|c} I & \boxed{\ } & \xi \\ \hline 0 & 0 & * \end{array}\right]$$

$y = Ax$

I

$$I \quad \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad \underline{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(more next week...)

Inverse Properties $A \in \mathbb{R}^{n \times n}$

- A invertible / nonsingular / nondegenerate
- A can be row reduced to I
- A can be col reduced to I $\times$
- A: n pivot positions x
- $\det(A) \neq 0$
- A has full rank (n)
- $Ax = 0 \Rightarrow x = 0$ (cols lin ind)
- $Ax = y \Rightarrow$ unique solution $x = A^{-1}y$ for every y
- cols of A lin ind
- cols of A span $\mathbb{R}^n$ $\Big\} \rightarrow$ cols of A form basis for $\mathbb{R}^n$
- rows of A lin ind
- rows of A span $\mathbb{R}^n$ $\Big\} \rightarrow$ rows of A basis for $\mathbb{R}^n$
- A^T is invertible
- A is a bijection
- 0 is not an eigenvalue of A
- A = finite product of elementary matrices

ea of these prop. gives you all the others

trying to invert a non invertible matrix is like dividing by 0

Algebraic Properties

- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$ for nonzero scalar k
- $(A^{-1})^T = (A^T)^{-1} = A^{-T}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ $A, B, C \in \mathbb{R}^{n \times n}$, invertible
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Note $(AB)^T = B^T A^T$

$[A^T][B] = [\;]^T$

$(A^T B)^T = B^T A$

$x^T y = y^T x$

$x \cdot y = y \cdot x$

Inverses Formula

$x^TAy = y^TA^Tx$

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{Adj}(A) \longrightarrow \text{Adjugate of } A$$

2x2 matrix

$$\longrightarrow A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Leftarrow \text{memorize}$$

Block Matrix Inversion LEMMA

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

$$\begin{matrix} y_1 \\ \cancel{y_2} \end{matrix} \begin{bmatrix} A & B \\ \cancel{C} & D \end{bmatrix} x$$

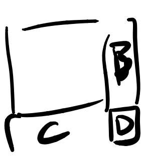
$$= \begin{bmatrix} (A-BD^{-1}C)^{-1} & (A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}-D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$

$\underline{D - CA^{-1}B}$ $\qquad$ $\underline{A - BD^{-1}C}$

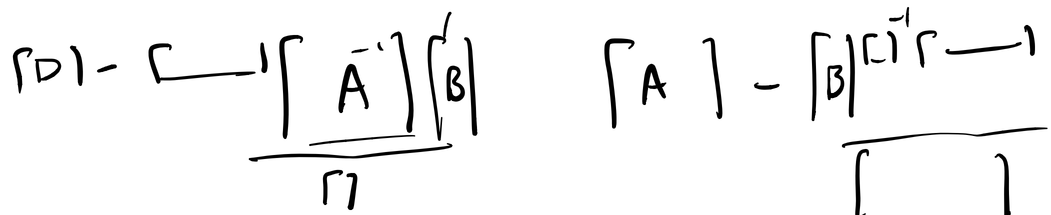
Schur complements of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

A, D square $\qquad$ A, $\underline{D - CA^{-1}B}$ invertible

or



D, $A - BD^{-1}C$ invertible



Woodbury Matrix Identity ← useful for practical implementations of KF

(Sherman Morrison Formula)

$$(A+B)^{-1} \neq A^{-1} + B^{-1} \qquad \frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$$

$$\boxed{[A + UCV]^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}}$$

$\underbrace{C^{-1} + VA^{-1}U}_{k \times k}$

- $A \in \mathbb{R}^{n\times n}$ invertible
- $U \in \mathbb{R}^{n \times k}$
- $C \in \mathbb{R}^{k \times k}$
- $V \in \mathbb{R}^{k \times n}$

$$[A] \pm [U][C][V]$$

Spars cols

C small

low rank

A know A^{-1}

$A + UCV$: add low rank component to A

want to update A^{-1}

$$A^{-1} - A^{-1}[U][C^{-1} + VA^{-1}U]^{-1}[V]A^{-1}$$

only have to invert $k \times k$

if $k = 1$ → scalar inverse

if $k=1$: Sherman Morrison formula

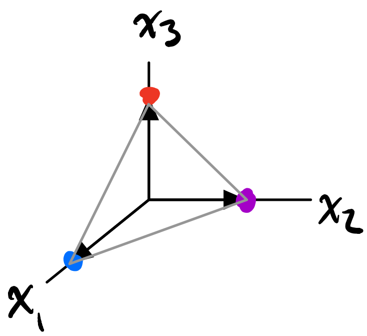
$$A + \begin{bmatrix} u \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \quad v \quad \end{bmatrix}$$

rank 1 :

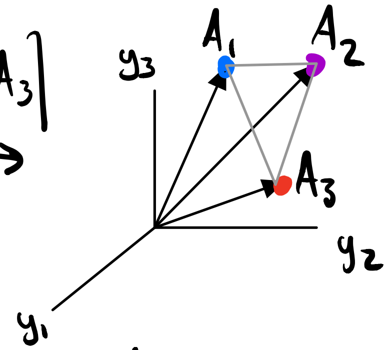
$$\underset{\substack{\downarrow \\ \text{col} \\ \text{vec}}}{u}\underset{\substack{\downarrow \\ \text{scalar}}}{C}\underset{\substack{\downarrow \\ \text{row} \\ \text{vec}}}{V} = \begin{bmatrix} u \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} | & & | \\ uv_1 & \cdots & uv_n \\ | & & | \end{bmatrix}$$

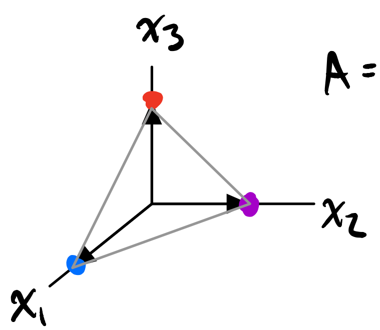
$$= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & & u_1 v_n \\ & & \\ u_n v_1 & & u_n v_n \end{bmatrix}$$

Eigenvalues & Eigenvectors: $A \in \mathbb{R}^{n \times n}$

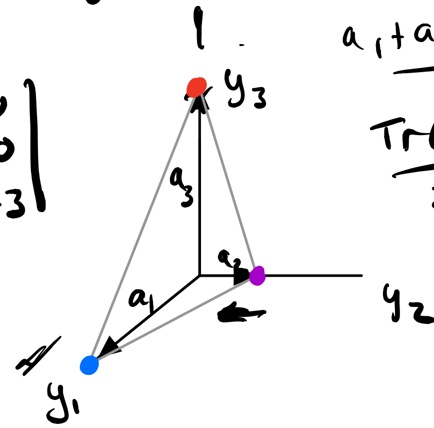


$A = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}$



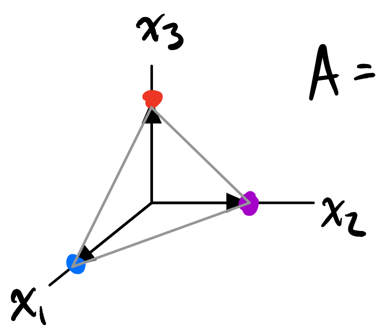


$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$



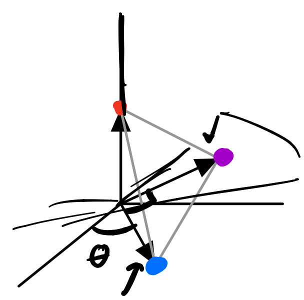
$\frac{a_1 + a_2 + a_3}{3}$

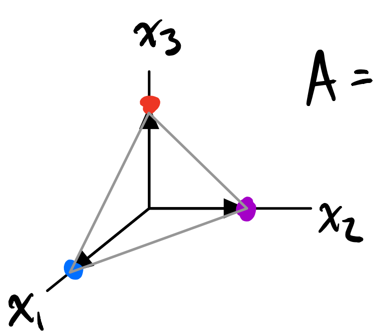
$\frac{Tr(A)}{3}$



$A = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

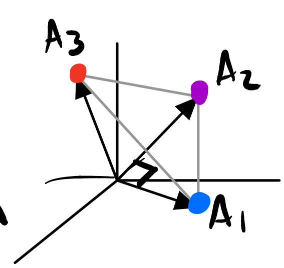
rotation around 3 axis

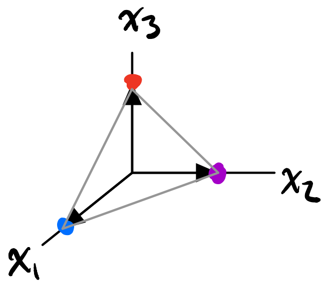




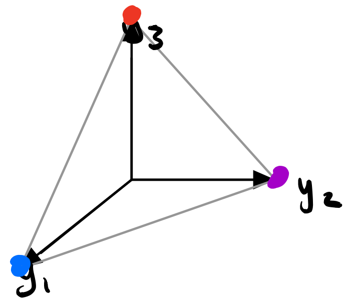
$A = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}$

general rotation





$A = \lambda I$



multiply by a number:

$\Rightarrow$ stretch / flip position on # line

multiply by a matrix

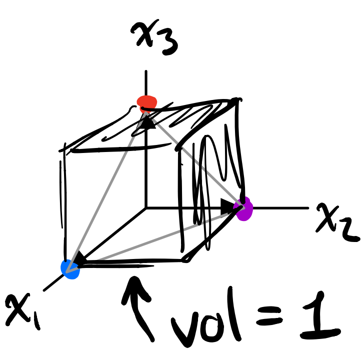
$\Rightarrow$ stretching / skewing / flipping / rotating your location in a vector space

Scalar valued functions used to measure how a matrix is transforming a space

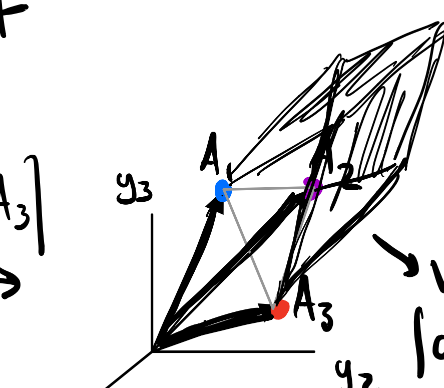
2 specifically...

"volume transformation"

- det(A): determinant



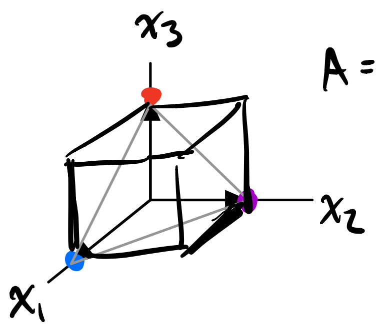
$A = [A_1 A_2 A_3]$



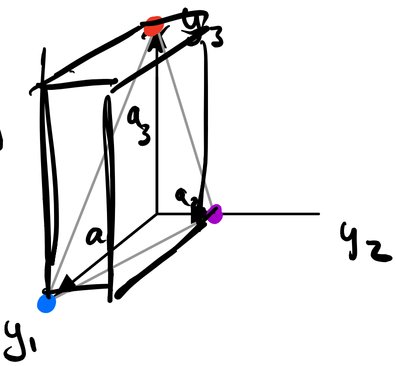
Vol = $|\det(A)|$

$\det(A) < 0$ means volume flipped inside out.

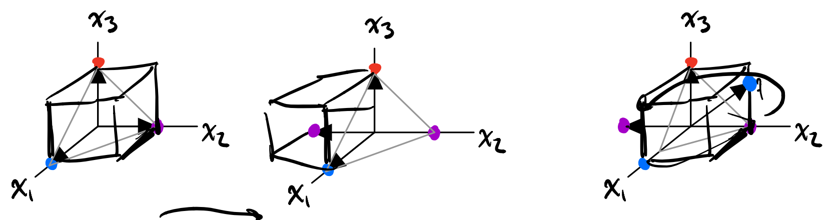
$\det(A) \Rightarrow$ degree n polynomial in the elements of A



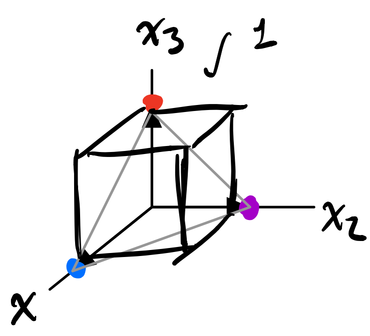
$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$



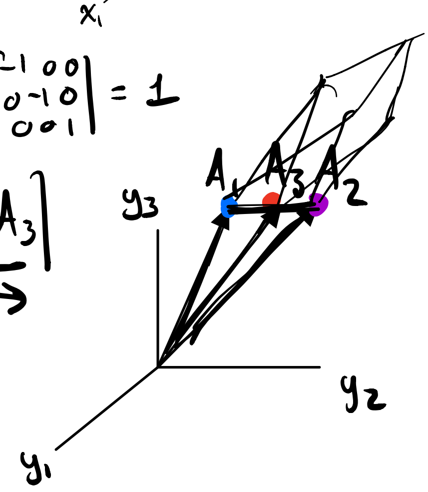
$$\underline{\det(A) = a_1 a_2 a_3}$$



$$\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \qquad \det \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$



$$A = \underline{[A_1 A_2 A_3]} \longrightarrow$$

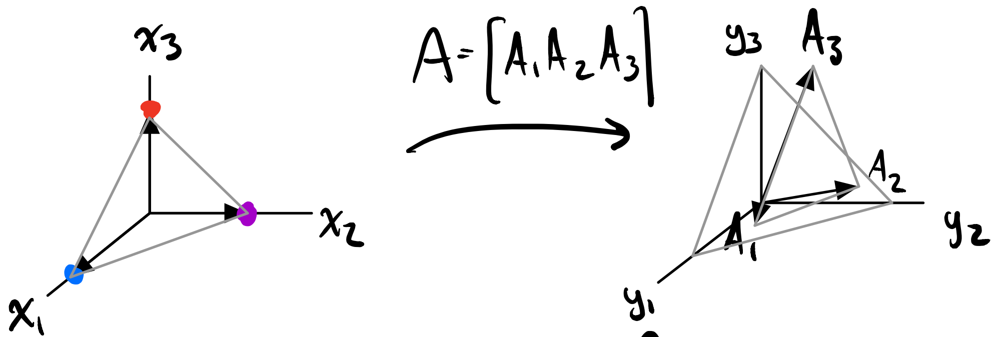


- $Tr(A)$: trace

  $Tr(A) = \sum_i A_{ii}$ → sum of diagonal elements

  $\frac{Tr(A)}{n}$: "average" amount the space gets stretched.

Eigenvalues / Eigenvectors



Eigenvector of A ; $v \in \mathbb{R}^n$

vector such that $Av = \lambda v$

scalar

Eigenvalue Eqn: $\lambda \in \mathbb{C}$

$$Av = \lambda v \qquad v \in \mathbb{C}^n$$

nonlinear eqn

because we have λv

if fix $\lambda \Rightarrow$ system of equations to solve for v.

$$\lambda v - Av = 0$$

$$\lambda I v - Av = 0$$

$$(\lambda I - A)v = 0 \Longrightarrow v \in N(\lambda I - A)$$

$\det(sI - A) = 0 \Rightarrow$ roots of $\det(sI - A)$ gives us λ.

$\underbrace{\det(sI - A)}_{\text{polynomial in } s}$

Characteristic Polynomial of A: $A \in \mathbb{R}^{n \times n}$

$$\chi_A(s) = \det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

solutions to $\chi_A(s) = 0 \Rightarrow$ eigenvalues

ie. λ is an eigenvalue then $\chi_A(\lambda) = 0$

$\Rightarrow \lambda I - A$ has a nontrivial nullspace

$\Rightarrow v$ is an eigenvector w/ eigenvalue λ if $(\lambda I - A)v = 0$ $\quad Av = \lambda v$

How many solns to $\chi_A(s) = 0$

Answer : n solutions

n "roots" of $\chi_A(s)$

Fundamental Thm of Algebra

$\Rightarrow n$ eigenvalues

if a matrix has some kind of rotation action

$\Rightarrow \chi_A(s)$ may have complex roots

Complex eigenvalues for $A \in \mathbb{R}^{n \times n}$ come in conjugate pairs

real matrix

$\lambda = a+bi$ (for $b \neq 0$) $\leftarrow$ if eigenvalue

$\bar{\lambda} = \lambda^* = a - bi$ $\leftarrow$ then eigenvalue

used to represent rotations.

corresponding pairs of eigenvectors give plane of rotation.

<u>Note</u>: length of eigenvector doesn't matter

$$A\underset{\sim}{v} = \lambda v \implies A\underline{\alpha v} = \lambda \alpha v$$

eigenvector is imprecise ...

$\boxed{\text{eigen subspace}}$ $\Leftarrow$ better description

Diagonalizing a Matrix:

$A \in \mathbb{R}^{n \times n}$ roots of $\chi_A(s) \Rightarrow$ eigenvalues

$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

find eigenvectors:

$$Av_i = \lambda_i v_i \qquad i = 1, \ldots, n$$

$$\begin{bmatrix} Av_1 & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$A\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$\Downarrow$

$$AP = PD$$

Assume P invertible

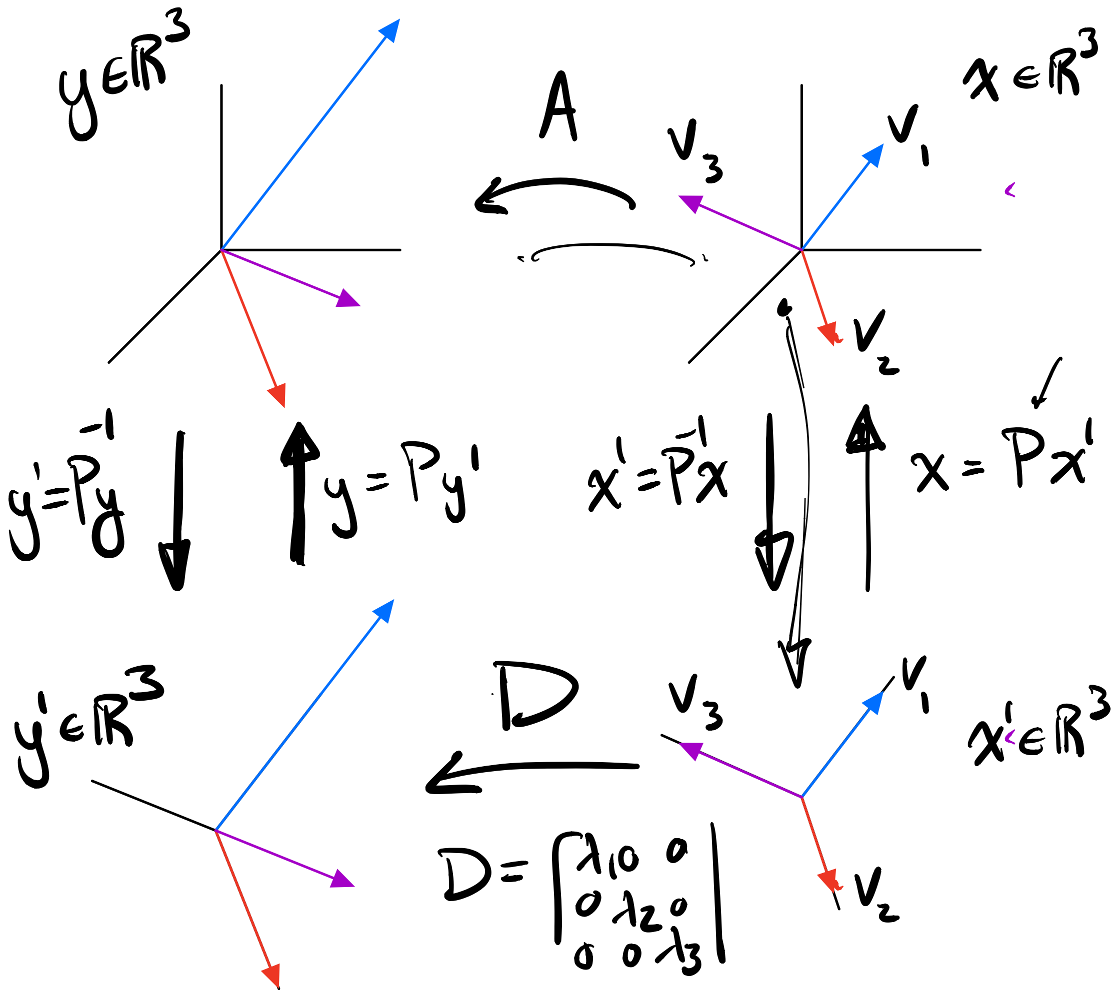
$$\underbrace{P^{-1}AP}_{\text{similarity transform}} = D \Leftarrow \text{called diagonalizing } A$$

$$\boxed{A = PDP^{-1}} \Leftarrow \text{diagonalization of } A$$

eigendecomposition

cols of P are eigenvectors

diagonal of D are eigenvalues



$$D = P^{-1} A P \qquad \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$A = \underbrace{P}_{\text{put back in original coords}} \underbrace{\left[D \right]}_{\text{scale ea. coord by eigenvalues}} \underbrace{P^{-1} x}_{\text{write } x \text{ in eigenvector basis}}$$

$$A^k = A \times \cdots \times A$$

$$= PDP^{-1} \times PDP^{-1} \times \cdots \times PDP^{-1}$$

$$= PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} P^{-1}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3$$

$$\dot{x} = Ax \longrightarrow x(t) = e^{At}x(0)$$

(more next week...)