Row Reduction / Rank (continuation)

Gauss Elimination

FROM LAST TIME $y = Ax$ ←

$[A\,|\,y] \underset{\boxed{\text{row reduce}}}{\sim} [I\,|\,x]$ for $A \in \mathbb{R}^{n\times n}$ invertible

same as left multiplying by a string of elementary matrices...

$E = E_k \cdots E_2 E_1$ ← "constructing A^{-1} slowly"

$E[v_1 \cdots v_n]$

$[Ev_1 \cdots Ev_n]$

$\underbrace{E^{-1}E}_{I}[A\,|\,y] = E^{-1}[\underbrace{EA}_{I}\,|\,\overset{x}{Ey}]$

if $E = A^{-1}$ $\quad = \underbrace{E^{-1}}_{A}[I\,|\,A^{-1}y] \rightarrow x = A^{-1}y$

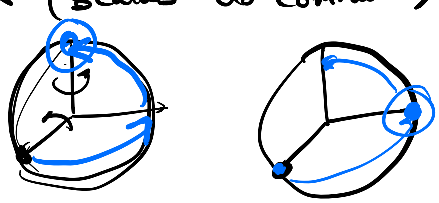
$[A\,|\,y] \quad = A[\underbrace{I}_{\text{coords of } A \text{ w.r.t cols of } A}\,|\,A^{-1}y]$ → $x = A^{-1}y$ is the coords of y w.r.t cols of A (which is x)

Left multiply by B → $By = BAx$

$AB \neq BA$ (matrices don't commute)

$ab = ba$ $\quad a, b \in \mathbb{R}$ (scalars do commute)

$R_1R_2 \neq R_2R_1$



Matrix Commutator of A & B ↘

$AB - BA$

Now A is not invertible ...

$A \in \mathbb{R}^{m \times n}$ $\qquad y = Ax$

3 cases trying to row reduce A to I and we fail

- $m < n$ full row rank (many solutions...)
- $m > n$ full col rank (no solution)
- not full col rank / not full row rank (many non-solutions...)

based on free variable v ...

1) $m < n$ full row rank

$$A = \begin{array}{c} \\ m \end{array}\begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \left[A_1 \mid A_2 \right] \end{array} \qquad y = Ax$$

$$E\left[A_1 \; A_2 \mid y\right]$$

construct E to be A_1^{-1}

$$\downarrow$$

$$\left[EA_1 \; EA_2 \mid Ey\right]$$

$$\rightarrow \left[\boxed{I} \mid A_1^{-1}A_2 \mid A_1^{-1}y\right]$$

specific solution

free variable

solution to $y = Ax \Rightarrow \boxed{\underline{x} = \begin{bmatrix} \boxed{A_1^{-1}y} \\ 0 \end{bmatrix} + \begin{bmatrix} A_1^{-1}A_2 \\ -I \end{bmatrix} v}$

cols span nullspace of A

$$Ax = \left[A_1 \; A_2\right]\begin{bmatrix} A_1^{-1}y \\ 0 \end{bmatrix} + \left[A_1 \; A_2\right]\begin{bmatrix} A_1^{-1}A_2 \\ -I \end{bmatrix} v$$

$$= A_1 A_1^{-1} y + A_2 0 + A_1 A_1^{-1} A_2 v - A_2 v$$

$$y = I \qquad I \, A_2 v - A_2 v$$

$\rightarrow [A \mid y] = E^{-1}E[A_1 \;\; A_2 \mid y]$

$= E^{-1}[EA_1 \;\; EA_2 \mid Ey]$

pick $E = A_1^{-1}$

$= A_1[I \;\; A_1^{-1}A_2 \mid A_1^{-1}y]$

cols of A_1 are a new basis for $\mathcal{R}(A)$

coords of A_1 w.r.t. cols of A_1

coords of A_2 w.r.t. cols of A_1

coords of y w.r.t cols of A_1

$y = A_1(A_1^{-1}y)$

---

2) $m > n$ full col rank (A tall)

$y = Ax$

$E^{-1}E[A \mid y] = E^{-1}[EA \mid Ey]$

$= E^{-1}\left[\begin{array}{c|c} I & Ey \\ 0 & \end{array}\right]$

$E^{-1} = [A \mid N]$

cols span $\mathcal{N}(A^T)$

$= [A \;\; N]\left[\begin{array}{c|c} I & [A \;\; N]^{-1}y \\ 0 & \end{array}\right]$

E^{-1}

$[A \;\; N]^{-1}y = \begin{bmatrix} z \\ z' \end{bmatrix}$

$= [A \;\; N]\left[\begin{array}{c|c} I & z \\ 0 & z' \end{array}\right]$

contradiction → no solutions

new basis

coords for A

coords for y

$\Rightarrow A = [A \;\; N]\begin{bmatrix} I \\ 0 \end{bmatrix}$ $\qquad y = [A \;\; N]\begin{bmatrix} z \\ z' \end{bmatrix} = Az + Nz'$

$$\rightarrow \quad \underline{y} = \underbrace{A\mathbf{z}}_{\text{piece of } y \text{ in } \mathcal{R}(A)} + \underbrace{N\mathbf{z}'}_{\text{piece of } y \text{ in } \mathcal{N}(A^T)} = [A \; N]\,\underline{[A \; N]^{-1} y}$$

$$y = \underline{A(A^TA)^{-1}A^Ty} + \underline{N(N^TN)^{-1}N^Ty}$$

Connection to LEAST SQUARES

$$[A \; N]^{-1} y = \begin{bmatrix} (A^TA)^{-1}A^T \\ (N^TN)^{-1}N^T \end{bmatrix} y = \begin{bmatrix} (A^TA)^{-1}A^Ty \\ (N^TN)^{-1}N^Ty \end{bmatrix}$$

$(A^TN = 0)$

$$x_1 = (A^TA)^{-1}A^Ty$$

$$\underline{x_2} = (N^TN)^{-1}N^Ty$$

x_1 is the LS solution to

$y = Ax$

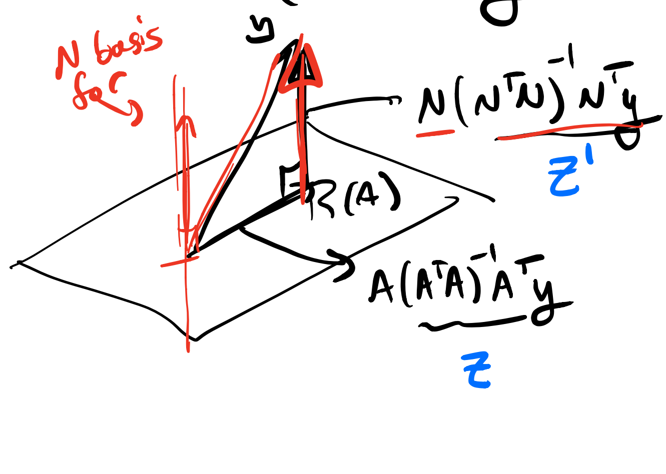
$y = Ax$

Component of y orthogonal to $\mathcal{R}(A)$

is $\underline{N\mathbf{z}'}$ ... $\underline{N(N^TN)^{-1}N^T}y$

projection of y orthogonal to $\mathcal{R}(A)$

$A\mathbf{z} = A(A^TA)^{-1}A^Ty \rightarrow$ projection of y onto the $\mathcal{R}(A)$



best we can do is choose

$$x = \mathbf{z} = \underline{(A^TA)^{-1}A^Ty}$$

least squares solution

Most general row reduction case ...

$$E^{-1}E[A_1\ A_2 | y] = E^{-1}[EA_1\ EA_2 | Ey] \qquad \begin{bmatrix} z \\ z' \end{bmatrix} = Ey$$

$$E^{-1} = [A_1\ N] \qquad = E^{-1}\left[\begin{array}{cc|c} I & B & z \\ 0 & \boxed{0} & z' \end{array}\right] \leftarrow \text{contradictions}$$

cols of N are a basis for $N(A_1^T)$

either A is too tall or rows are lin dep.

$$[A | y] \sim [I | x]$$

A_1 is not invertible, but cols of A_2 are linear dep on A_1 ...
(A_1 is tall)

ie. $A_2 = A_1 B$

B → coords of A_2 w.r.t cols of A_1

A_1 cols are basis for $\mathcal{R}(A)$

$$\longrightarrow \quad = [A_1\ N]\left[\begin{array}{c|c|c} I & B & z \\ 0 & 0 & z' \end{array}\right]$$

new basis for codomain

coords of A_1 w.r.t new basis

coords of A_2 w.r.t new basis

coords of y w.r.t. new basis

$$A_1 = [A_1\ N]\begin{bmatrix} I \\ 0 \end{bmatrix} \qquad A_2 = [A_1\ N]\begin{bmatrix} B \\ 0 \end{bmatrix} \qquad y = [A_1\ N]\begin{bmatrix} z \\ z' \end{bmatrix}$$

$$y = \underbrace{A_1 z}_{\substack{\text{comp of} \\ y \text{ in } \mathcal{R}(A)}} + \underbrace{N z'}_{\substack{\text{comp of} \\ y \text{ orthogonal} \\ \text{to } \mathcal{R}(A)}}$$

choosing $x = x_1$ gets us as close to y as possible...

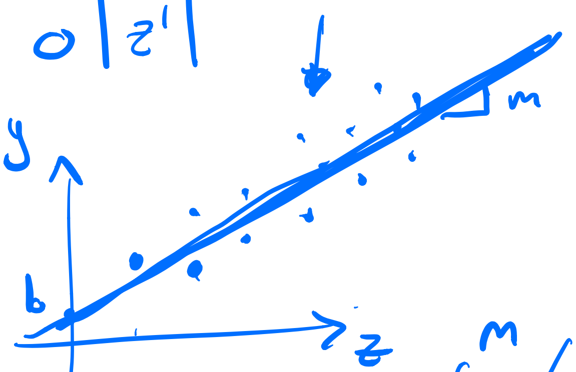
$$\left[\, A \mid y \,\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & \cdots & x_n \end{array} \\ \left[\, I \mid A^{-1} y \,\right] \end{array}$$

$$\begin{array}{c} x_1 \cdots x_n \\ \left[\begin{array}{c|c} I & z \\ 0 & z' \end{array}\right] \end{array}$$

Side note

For tall A...

Model: $y = \underline{m} z + b$



$$y_1 = [z_1 \;\; 1] \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{x}$$

$$y_2 = [z_2 \;\; 1]$$

$$\vdots$$

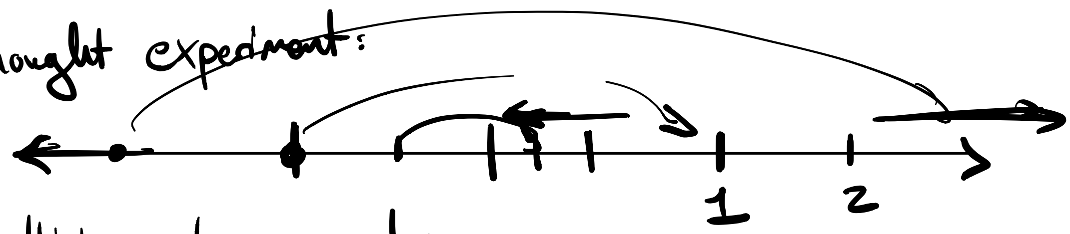
$$y_T = [z_T \;\; 1]$$

$$x = \begin{bmatrix} m \\ b \end{bmatrix} \qquad A = \begin{array}{cc} m & b \\ \multicolumn{2}{c}{\begin{bmatrix} z_1 & 1 \\ z_2 & 1 \\ \vdots & \\ z_T & 1 \end{bmatrix}} \end{array}$$

$$Ax = A \begin{bmatrix} m \\ b \end{bmatrix}$$

Complex #'s:

Thought experiment:



multiplying by a number stretches your position on the # line

z^2

$z = 1 \longrightarrow 1$

$z = 2 \longrightarrow 4$

$z = \frac{1}{2} \longrightarrow \frac{1}{4}$

$z = -1 \longrightarrow$

$z = -2$

$z = -\frac{1}{2} \longrightarrow \frac{1}{4}$

square root of z is

m s.t. $m^2 = z$

for positive #s

$\sqrt{4} = 2$

$\sqrt{1} = 1$

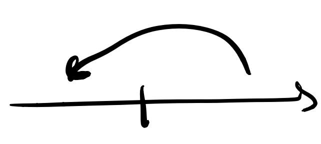
$\sqrt{1/4} = 1/2$

for negative #'s

what is "half" of a flip from one side of the # line to the other...

$-1 = m \times m$

$m = i$



complex #'s come from the idea of trying to break down flipping sides of the # line into smaller pieces

complex #'s are closely rotated to rotations. →

Euler's Formula : $e^{i\pi} + 1 = 0$

the exponential of an imaginary # is closely related to an oscillation → $e^{i\theta} = \cos\theta + \sin\theta\, i$

proof by Taylor expansion

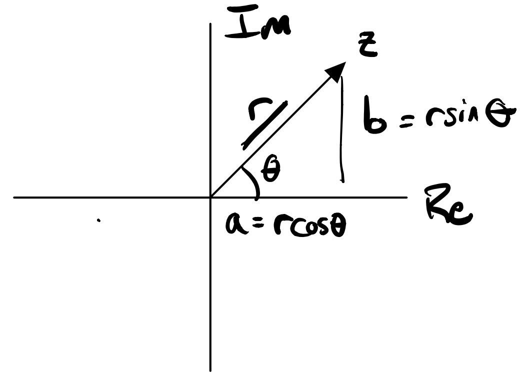
Representing rotations → complex #'s/quaternion
→ rotation matrix

↳ eigenvalues are complex.

complex = $a + bi$ ($i = \sqrt{-1}$)

quaternion = $a + bi + cj + dk$ → 4D complex #

Complex Plane



$z \in \mathbb{C}$

2 representations

- Cartesian :

  $z = a + bi$

- polar : $r > 0$

  $z = re^{i\theta} = r\cos\theta + r\sin\theta i$

  $e^{i\theta} = \cos\theta + \sin\theta i$

  $|z| = \sqrt{a^2+b^2} = r$

other way: $r = \sqrt{a^2+b^2}$ $\quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$

Adding : $z_1 = a_1 + b_1 i$
$z_2 = a_2 + b_2 i$

$z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)i$

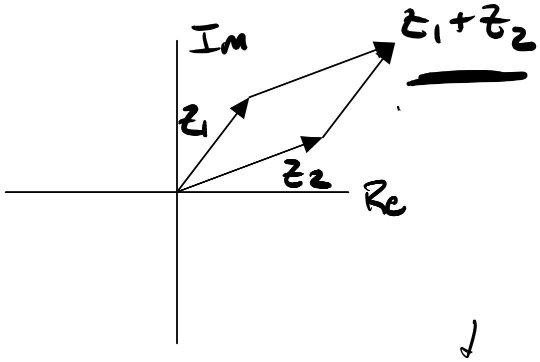
Multiplying

$$\left.\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + b_1 b_2 i^2 + (a_1 b_2 + a_2 b_1)i \\ &= a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i \end{aligned}\right\} \text{Cartesian}$$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

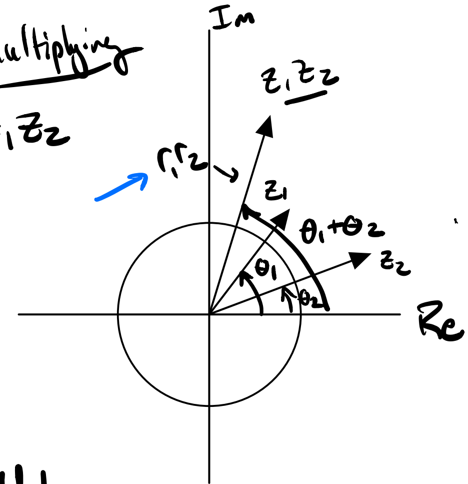
$\Rightarrow$ magnitudes multiply, phases add.

Adding



Multiplying

$z_1 z_2$



Conjugation

$z = a+bi = re^{i\theta}$

$z^* = \bar{z} = a - bi = re^{-i\theta}$

$*$ is related to transpose

$A \in \mathbb{C}^{m\times n}$

A^*: the conjugate transpose of A

→ transpose A

→ conjugate all complex #s

if $A \in \mathbb{R}^{m\times n}$ then $A^* = A^T$

Note

if z_1 & z_2 outside unit circle, z_1z_2 farther away from 0

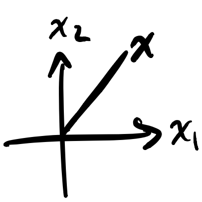
if z_1 & z_2 inside unit circle z_1z_2 closer to 0

Matlab:

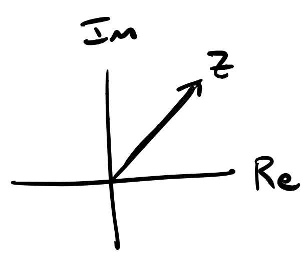
→ transpose (A)

→ ctranspose (A)

A' = ctranspose(A)



$|x| = (x^T x)^{1/2}$



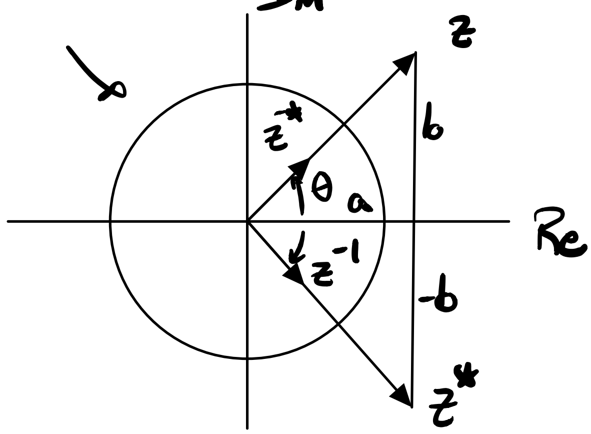
$|z| = (z^* z)^{1/2}$

$(a-bi)(a+bi)$

$a^2 + b^2 \cancel{i^2}$

$= (a^2+b^2)^{1/2}$

More diagrams z, z^*, z^{-1}, z^{-*}



$z = a + bi = re^{i\theta}$

$\to z^* = a - bi = re^{-i\theta}$

$\to z^{-1} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$

$z^{-*} = \frac{1}{re^{-i\theta}} = \frac{1}{r} e^{i\theta}$

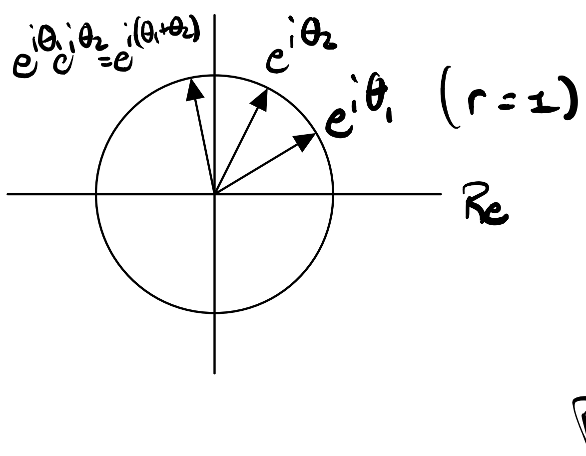
Interesting Analogies

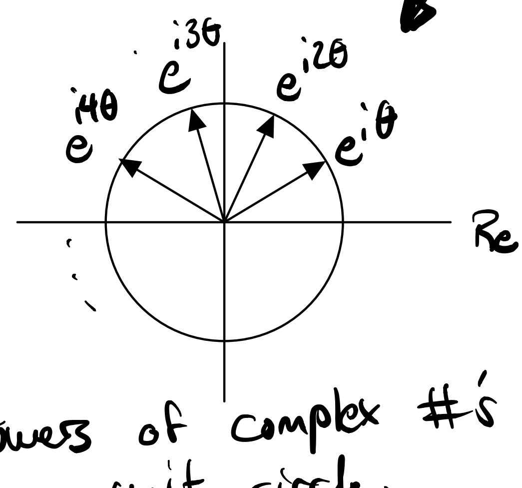
$z \sim A \quad z^* \sim A^T \quad z^{-1} \sim A^{-1} \quad z^{-*} \sim A^{-T}$

(polar decomposition of a matrix...)

Complex #'s w magnitude 1.

$|z| = 1$ $\quad (e^{i\theta})^2$





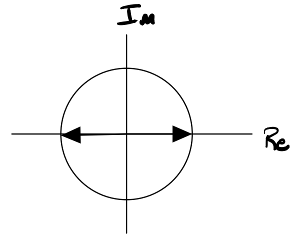
powers of complex #'s
on unit circle
can be used to represent

Roots of Unity

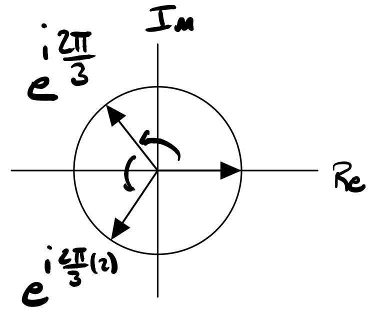
rotating in steps

solutions to eqn: $z^n = 1$

$z^2 = 1 \Longrightarrow z = 1, -1$



$z^3 = 1 \Longrightarrow z = 1, e^{i\frac{2\pi}{3}}, e^{i\frac{2\pi}{3}2}$



$\left(e^{i\frac{2\pi}{\cancel{3}}}\right)^{\cancel{3}} = e^{i2\pi} = 1$ goes around once

$\left(e^{i\frac{2\pi}{3}2}\right)^3 = e^{i4\pi} = 1$ goes around twice...

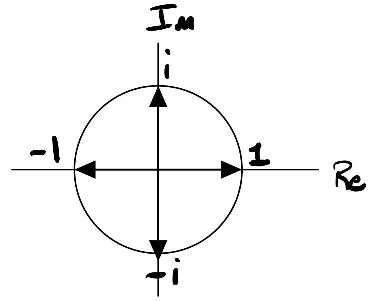
$e^{i\frac{2\pi}{3}2} = e^{-i\frac{2\pi}{3}}$

$z = 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{2\pi}{3}2} = e^{-i\frac{2\pi}{3}}$

rotation twice around

rotating backwards once around

$z^4 = 1 \Rightarrow z = 1, i, -1, -i$



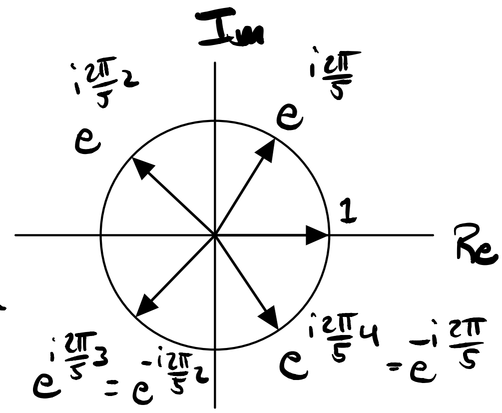
$1^4 = 1$

$(i)^4 = (-1)i^2 = -i\,i = 1$

$$z^4 = 1 \Rightarrow z = 1, e^{i\pi/2}, e^{i\pi} = e^{-i\pi}, e^{i\frac{3\pi}{2}} = e^{-i\frac{\pi}{2}}$$

$$z^5 = 1 \Rightarrow .$$

$$z = 1, e^{i\frac{2\pi}{5}}, e^{i\frac{2\pi}{5}2}, e^{i\frac{2\pi}{5}3}, e^{i\frac{2\pi}{5}4}$$



$$z^n = 1 \Rightarrow$$

# of times you go around the circle to get back to 1

$$\left[\, z = e^{i\frac{2\pi}{n}k} \quad k = 0, 1, \dots, n-1 \right| n, n+1$$

$$k = 0, -n+1, \dots, -1$$

Fundamental Building blocks of DISCRETE FOURIER TRANSFORM

heart of digital signal processing $\leftarrow$ DFT

$k = n$

$$e^{i\frac{2\pi}{\cancel{n}}\cancel{n}} = 1 \cdot e^{i\frac{2\pi}{n}0}$$

$$e^{i\frac{2\pi}{n}(n+1)} = \underbrace{e^{i\frac{2\pi n}{n}}}_{1} e^{i\frac{2\pi}{n}1} = e^{i\frac{2\pi}{n}} \quad k = 1$$

$$z(t) = \left(e^{i\frac{2\pi}{n}k}\right)^t$$

time

sampling period

frequency

Eigenvalues / Eigenvectors:

Right eigenvectors (eigenvalues) $A \in \mathbb{R}^{n \times n}$

$$Av = \lambda v$$

$v \in \mathbb{C}^n \to$ eigenvector

eigensubspace

$\lambda \in \mathbb{C} \to$ eigenvalue

Left eigenvectors

$$w^T A = w^T \lambda \quad w \in \mathbb{C}^n$$

$$\lambda \in \mathbb{C}$$

every eigenvalue λ_i has a right eigenvector v_i

left eigenvector w_i^T

If we have a basis of right eigenvectors $\begin{bmatrix} v_1 \cdots v_n \end{bmatrix} = P$

$$AP = PD \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\underbrace{A = PDP^{-1}}_{\text{diagonalization of } A}$$

basis of left eigenvectors... $Q = \begin{bmatrix} -w_1^T- \\ \vdots \\ -w_n^T- \end{bmatrix}$

$$QA = DQ \Rightarrow A = \overbrace{Q^{-1}DQ}^{\text{diagonalization}}$$

$P^{-1} = Q$ (almost) ★ $\Longleftarrow$ ★

$A = PDP^{-1}$

$= \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{\substack{\text{right} \\ \text{eigenvectors}}} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\text{eigenvalues}} \underbrace{\begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}}_{\text{left eigenvectors}}$

$= \sum_i \lambda_i \underbrace{v_i w_i^T}_{\begin{bmatrix} v_i \end{bmatrix} \begin{bmatrix} - & w_i^T & - \end{bmatrix}}$ $\longleftarrow$ dyadic expansion

FACT: if A is diagonalizable

right eigenvector $v_i^T w_j = 0$ for $j \neq i$

left eigenvector $w_i^T v_j = 0$ for $j \neq i$

Summary: ★

$A v_i = P D P^{-1} v_i = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} v_i}$

$=$

$= \lambda_i v_i \quad \begin{bmatrix} 0 \\ \lambda_i \\ 0 \end{bmatrix} \leftarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Ax

$x^T A$

Spectral Mapping Thm: $A \in \mathbb{R}^{n \times n}$

$f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ f: polynomial (analytic)

$$f(A) = \alpha_n A^n + \alpha_{n-1} A^{n-1} + \alpha_{n-2} A^{n-2} + \cdots$$

$$A^k = \overbrace{A \times A \times \cdots \times A}^{k \text{ times}}$$

$$= PDP^{-1} \times PDP^{-1} \times PDP^{-1} \times \cdots \times PDP^{-1}$$

$$= PD^k P^{-1} = P \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} P^{-1}$$

$$\alpha_1 A^{k_1} + \alpha_2 A^{k_2} = \alpha_1 PD^{k_1}P^{-1} + \alpha_2 PD^{k_2}P^{-1}$$

$$= P(\alpha_1 D^{k_1} + \alpha_2 D^{k_2})P^{-1}$$

$$= P \begin{bmatrix} \alpha_1 \lambda_1^{k_1} + \alpha_2 \lambda_1^{k_2} & & \\ & \ddots & \\ & & \alpha_1 \lambda_n^{k_1} + \alpha_2 \lambda_n^{k_2} \end{bmatrix} P^{-1}$$

for any f like above...

$$f(A) = Pf(D)P^{-1} = P \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} P^{-1}$$

Summary:

the eigenvectors of $f(A)$ are the eigenvectors of A

if λ_i is eigenvalue of A then $f(\lambda_i)$ is eigenvalue of $f(A)$

Spectrum of A is the set of eigenvalues of A.

$$\rho(A) = \{\lambda_1, \dots, \lambda_n\}$$

Finding eigenvalues

$$Av = \lambda v$$

roots of $\chi_A(s) = \det(sI - A)$ $\quad (\lambda I - A)v = 0$

$\chi_A(s)$: characteristic polynomial

$$\chi_A(s) = \det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1 s + \alpha_0$$

solve $\chi_A(s) = 0 \implies$ n roots $\lambda_1, \dots, \lambda_n$

$$\chi_A(\lambda_1) = 0$$

$$\vdots$$

$$\chi_A(\lambda_n) = 0$$

even for real A, λ_i might be complex

if A real the complex eigenvalues come in conjugate pairs

$A \in \mathbb{R}^{n \times n}$

$\underline{Im(\lambda) \neq 0} \Rightarrow \lambda, \lambda^* \leftarrow$ both eigenvalues

Cayley <u>Hamilton</u> :

$$\chi_A(A) = 0$$

$$= A^n + \alpha_{n-1} A^{n-1} + \dots + \alpha_1 A + \alpha_0 I = 0$$

Matrix of 0's

Proof based on spectral mapping thm

if A is diagonalizable...

$$\chi_A(A) = PD^nP^{-1} + \alpha_{n-1} PD^{n-1}P^{-1} + \dots + \alpha_1 PDP^{-1} + \alpha_0 I$$

PP^{-1}

$$= P\left[D^n + \alpha_{n-1} D^{n-1} + \dots + \alpha_1 D + \alpha_0 I\right]P^{-1}$$

$$= P \begin{bmatrix} \cancel{\chi_A(\lambda_1)}^{\,0} & & \\ & \ddots & \\ & & \cancel{\chi_A(\lambda_n)}^{\,0} \end{bmatrix} P^{-1} = 0$$

$$\Rightarrow A^n + \alpha_{n-1} A^{n-1} + \dots + \alpha_1 A + \alpha_0 I = 0$$

$$A^n = \underline{-\alpha_{n-1} A^{n-1} -} \dots \underline{-\alpha_1 A - \alpha_0 I}$$

any polynomial function of A of any degree $\longrightarrow$ can be rewritten w/ degree $\underline{n-1}$

Ex.

$$A^{2n} = A^n A^n = (**) \leftarrow \text{deg } 2(n-1)$$

A^n

$A^6 \quad A^3 A^3 \leftarrow \quad A^4 \quad A^3 A \leftarrow$

---

if A is invertible...

$$\left(A^n + \alpha_{n-1} A^{n-1} + \dots + \alpha_1 A + \alpha_0 I\right) A^{-1} = 0 A^{-1}$$

$$A^{n-1} + \alpha_{n-1} A^{n-2} + \dots + \alpha_1 I + \alpha_0 A^{-1} = 0$$

$$A^{-1} = -\frac{1}{\alpha_0} \underline{A^{n-1}} - \frac{\alpha_{n-1}}{\alpha_0} A^{n-2} - \dots - \frac{\alpha_1}{\alpha_0} I$$

slick

How do eigenvectors change under similarity transforms? $P = [v_1 \cdots v_n]$

Say I have $\underline{A} = \underline{PDP^{-1}}$ ←

→ similarity transform $\underline{MAM^{-1}}$

what are the eigenvectors of MAM^{-1}

$$MAM^{-1} = \underbrace{MP}_{\text{cols are right evecs}} D \underbrace{P^{-1}M^{-1}}_{\text{rows are left evecs}}$$

Eigenvalues stay the same under a similarity transform.

AB, BA have same evals A, B invertible

$\underline{AB} = B^{-1}(\underline{BA})B$ ← related by a similarity transform

$B^{-1}(BA)B = AB$

Det: volume change

Trace: sum of diagonal elements

$$\det(A) = \prod_i \lambda_i \qquad \bigstar$$

$$\mathrm{Tr}(A) = \sum_i \lambda_i$$

$$\left.\begin{aligned} \det(AB) &= \det(A)\det(B) \\ \det(P^{-1}) &= \frac{1}{\det(P)} \\ \mathrm{Tr}(AB) &= \mathrm{Tr}(BA) \end{aligned}\right\} \text{properties}$$

$$\begin{aligned} \det(A) &= \det(PDP^{-1}) \\ &= \det(P)\det(D)\det(P^{-1}) \\ &= \cancel{\det(P)}\,\underline{\det(D)}\,\frac{1}{\cancel{\det(P)}} \\ &= \prod_i \lambda_i \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}(A) = \mathrm{Tr}(PDP^{-1}) &= \mathrm{Tr}(DP^{-1}P) \\ &= \mathrm{Tr}(D) = \sum_i \lambda_i \end{aligned}$$